Adaptive basket liquidation

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Abstract

We consider the infinite time-horizon optimal basket portfolio liquidation problem for a von Neumann-Morgenstern investor in a multi-asset extension of the liquidity model of Almgren (2003) with cross-asset impact. Using a stochastic control approach, we establish a "separation theorem": the sequence of portfolios held during an optimal liquidation depends only on the (co-)variance and (cross-asset) market impact of the assets, while the speed with which these portfolios are attained depends only on the utility function of the trader. We derive partial differential equations for both the sequence of attained portfolios and the trading speed.

Keywords: optimal basket liquidation, dynamic trading strategies, separation theorem, utility maximization
JEL codes: G10, G12, G14, G20, G33

1 Introduction

Investors frequently wish to trade several assets simultaneously. For example, rebalancing an index tracking fund may require trading in several hundred different shares. Optimal execution of such a basket trade depends not only on the (co-)variances of the assets, but also on the (cross-asset) price impact of trading. Our goal in this paper is to determine the utility maximizing trading strategy for such *basket liquidations* with respect to a wide range of utility functions and describe it as the solution to a partial differential equation. We find that the set of portfolios that are held during the liquidation is independent of the investor's utility function but only depends on the market volatility and liquidity structure. The utility function only influences how quickly the investor executes the trades.

For practical applications, we can determine the utility maximizing trading strategy by executing two steps. First, we derive the deterministic mean-variance optimal basket trading strategy. While we show that such a strategy always exists, finding it numerically can be challenging due to the high number of dimensions. Second, we solve a partial differential equation and obtain an optimal "relative trading speed". This PDE depends only on the risk aversion of the utility function, but not on the market parameters such as the covariance structure. Under the utility maximizing trading strategy, the portfolio evolves exactly as in the mean-variance optimal trading strategy, but with a time transformation given by the relative trading speed. This establishes a "separation theorem" for optimal liquidation: Investors with different risk attitudes will choose the same basket liquidation strategy, but execute it at a different speed. Because of this separation, utility maximization becomes a numerically tractable option for implementing adaptive basket liquidation strategies in practice.

We consider a continuous-time, infinite time-horizon *multiple asset* extension of the model introduced by Almgren and Chriss (2001) and Almgren (2003). In particular, we allow for non-linear cross-asset price impacts. However, we need to assume that price impact scales like a power law, i.e., that trading a times faster results in a price impact multiplied by a^α where $\alpha > 0$ is a constant. In this market model, we first show that a unique mean-variance optimal trading strategy exists and that it satisfies both Bellman's principle of optimality and the Beltrami identity. Furthermore, the mean-variance costs of liquidation fulfil the dynamic programming PDE. Thereafter, we construct a solution to the HJB equation for utility maximization. The key observation is that the expected utility under optimal adaptive liquidation is identical for different portfolios with the same mean-variance cost of deterministic execution. We can therefore construct the utility maximization value function by

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solving two-dimensional PDEs instead of high-dimensional PDEs. Finally we apply a verification argument to show that the solution to the HJB equation is indeed the value function.

Building on empirical investigations of the market impact of large transactions, a number of theoretical models of illiquid markets have emerged. One part of these models focuses on the underlying mechanisms for illiquidity effects, e.g., Kyle (1985) and Easley and O'Hara (1987). We follow a second line that takes the liquidity effects as given and derives optimal trading strategies within such a stylized model market. Several market models have been proposed for this purpose, e.g., Bertsimas and Lo (1998), Almgren and Chriss (2001), Almgren (2003), Obizhaeva and Wang (2006) and Alfonsi, Fruth, and Schied (2010). While the advantages and disadvantages of these models are still a topic of ongoing research, we apply an n-dimensional extension of the market model introduced by Almgren (2003) in this paper for the following reasons. First, it captures both the permanent and temporary price impacts of large trades, while being sufficiently simple to allow for a mathematical analysis. It has thus become the basis of several theoretical studies, e.g,. Rogers and Singh (2010), Almgren and Lorenz (2007), Carlin, Lobo, and Viswanathan (2007) and Schöneborn and Schied (2009). Second, it demonstrated reasonable properties in real world applications and serves as the basis of many optimal execution methodologies run by practitioners (see e.g., Kissell and Glantz (2003), Schack (2004), Abramowitz (2006), Simmonds (2007) and Leinweber (2007)).

Within the optimal liquidation literature, most research was directed to finding the optimal *deterministic* or *static* liquidation strategy[1]. Only recently, academic research has started to investigate the optimization potential of adaptive strategies. Both Almgren and Lorenz (2007) and Schied and Schöneborn (2009) consider the special case of a single asset with linear price impact. Almgren and Lorenz (2007) analyse mean-variance traders in this setting and find that these always follow aggressive-in-the-money strategies, i.e., they accelerate trading after beneficial price moves and slow trading down after adverse price movements. Schied and Schöneborn (2009) derive optimal trading strategies for utility maximizing investors and can explain both aggressive-in-the-money and passive-in-the-money strategies. Their solution is a special case of the solution presented in this paper, however derived in a different way: They directly derive the optimal adaptive trading strategy without referring to mean-variance optimization. To our knowledge, the only paper so far considering adaptive liquidation of baskets is Schied, Schöneborn, and Tehranchi (2010). They limit the analysis to utility maximizing investors with an exponential (CARA) utility function and find that adaptive strategies offer no optimization benefit for this class of investors irrespective of any finite liquidation time horizon. For an infinite time horizon, we obtain this finding as a special case.

The rest of this paper is structured as follows. In Section 2, we introduce the multiple asset market model. The investor's trading target is discussed in Section 3. Thereafter, we first show in Section 4 that an optimal deterministic strategy exists for mean-variance optimization and subsequently use this strategy to construct the optimal strategy for utility maximization in Section 5. All proofs are given in Appendix A.

2 Market model

We assume that there are $n \geq 1$ risky assets and a risk-free asset traded. In this market, we consider a large investor who needs to liquidate a basket portfolio $\mathrm{X}_0 = (X_0^1, \dots, X_0^n) \in \mathbb{R}^n$ of shares in the n risky assets by time $T > 0$. The investor chooses a liquidation strategy that we describe by the portfolio $\mathrm{X}_t \in \mathbb{R}^n$ held at time t and that satisfies the boundary condition $\mathrm{X}_T = 0$. We assume that $t \mapsto \mathrm{X}_t$ is absolutely continuous with derivative $\dot{\mathrm{X}}_t =: -\xi_t$, i.e.,

$$\mathrm{X}_t = \mathrm{X}_0 - \int_0^t \xi_s \, ds.$$

For questions such as hedging derivatives, the restriction to absolutely continuous strategies is severe, since it excludes for example the Black-Scholes hedging strategy. For an analysis of optimal liquidation strategies, the restriction appears less grave, since reasonable optimal strategies can be expected to have bounded variation.[2]

Due to insufficient liquidity, the investor's trading rate ξ_t is moving the market prices. We consider an n-dimensional extension of the model introduced by Almgren (2003) (see also Bertsimas and Lo (1998), Almgren

[1] Notable exceptions describing optimal adaptive strategies include Subramanian and Jarrow (2001), He and Mamaysky (2005), Almgren and Lorenz (2007) and Çetin and Rogers (2007).

[2] Nevertheless, it would be desirable to allow block trades, i.e., jumps in X_t. Analyses of models that allow for such block trades (e.g., Obizhaeva and Wang (2006) and Alfonsi, Fruth, and Schied (2010)) reveal that for realistic parameters the optimal trading strategy is absolutely continuous except for very small block trades at the beginning and end of trading. Numerically, the optimal strategy is almost unchanged by the provision of block trades. Unfortunately, allowing for block trades significantly complicates the mathematical analysis. We therefore believe that it is acceptable to limit the discussion to absolutely continuous strategies.

and Chriss (1999) and Almgren and Chriss (2001) for discrete-time precursors of this model). The transaction price vector $\mathrm{P}_t \in \mathbb{R}^n$ in this model is the difference of the fundamental price $\tilde{\mathrm{P}}_t \in \mathbb{R}^n$ and the price impact $\mathrm{Imp}_t((\xi_s)_{0\le s\le t}) \in \mathbb{R}^n$:

$$\mathrm{P}_t = \tilde{\mathrm{P}}_t - \mathrm{Imp}_t((\xi_s)_{0\le s\le t}).$$

The multi-asset price impact Imp_t is assumed to be of the following special form:

$$\mathrm{Imp}_t((\xi_s)_{0\le s\le t}) := \int_0^t \mathrm{PermImp}(\xi_s)ds + \mathrm{TempImp}(\xi_t) \in \mathbb{R}^n.$$

The incremental order ξ_t therefore induces both a *permanent* and a *temporary impact* on market prices. The permanent impact $\mathrm{PermImp} : \mathbb{R}^n \to \mathbb{R}^n$ accumulates over time and is assumed to be linear:

$$\mathrm{PermImp}(\xi) := \Gamma\xi$$

where $\Gamma = (\Gamma^{ij}) \in \mathbb{R}^{n\times n}$ is a symmetric $n \times n$ matrix. Linearity and symmetry of the permanent impact are necessary to rule out quasi-arbitrage opportunities as was observed by Huberman and Stanzl (2004). The temporary impact $\mathrm{TempImp} : \mathbb{R}^n \to \mathbb{R}^n$ vanishes instantaneously and thus only affects the incremental order ξ_t itself. It is a possibly non-linear function. The idealization of instantaneous recovery of the temporary impact is derived from the well-known resilience of stock prices after order placement. It approximates reality reasonably well as long as the time intervals between the physical placement of orders are longer than a few minutes. See, e.g., Bouchaud, Gefen, Potters, and Wyart (2004), Potters and Bouchaud (2003), and Weber and Rosenow (2005) for empirical studies on resilience in order books and Obizhaeva and Wang (2006), Alfonsi, Fruth, and Schied (2008) and Alfonsi, Fruth, and Schied (2010) for corresponding market impact models.

This model provides a high degree of analytical tractability while still being sufficiently flexible to capture the relevant aspects of both the permanent and temporary price impacts of large trades. It is to our knowledge the only liquidity model that has become the basis of theoretical studies not only on the topic it was designed for (optimal portfolio liquidation), but also on several other topics such as hedging (Rogers and Singh (2010)), investment decision and implementation (Engle and Ferstenberg (2007)) and on the interaction of market participants in illiquid markets (Carlin, Lobo, and Viswanathan (2007), Schöneborn and Schied (2009)). Furthermore, it demonstrated reasonable properties in real world applications.

When the large investor is not active, it is assumed that the fundamental price process $\tilde{\mathrm{P}}$ follows an n-dimensional Bachelier model. The resulting vector-valued transaction price dynamics are hence given by

$$\mathrm{P}_t = \tilde{\mathrm{P}}_0 + \sigma \mathrm{B}_t + \Gamma(\mathrm{X}_t - \mathrm{X}_0) - \mathrm{TempImp}(\xi_t).$$

Equivalently, the transaction price for the i^{th} asset is given by

$$P_t^i = \tilde{P}_0^i + \sum_{j=1}^n \sigma^{ij} B_t^j + \sum_{j=1}^n \Gamma^{ij}(X_t^j - X_0^j) - \mathrm{TempImp}^i(\xi_t);$$

for an initial fundamental price vector $\tilde{\mathrm{P}}_0 \in \mathbb{R}^n$, a standard n-dimensional Brownian motion B starting at $\mathrm{B}_0 = 0$, and a $n \times n$ volatility matrix $\sigma = (\sigma^{ij}) \in \mathbb{R}^{n\times n}$. We assume that σ is non-degenerate with covariance matrix $\Sigma := \sigma\sigma^\top \in \mathbb{R}^{n\times n}$. At first sight, it might seem to be a shortcoming of this model that it allows for negative asset prices. But on the scale we are considering, the price process is a random walk on an equidistant lattice and thus perhaps better approximated by an arithmetic rather than, e.g., a geometric Brownian motion.

Note that we assumed that the stock has no drift. Almgren and Chriss (2001) found that if the temporary impact is linear, then the effect of a non-zero drift can be separated from the problem of optimal liquidation. More precisely, the optimal strategy in a market with drift is the sum of two strategies. The first of these strategies is the optimal liquidation strategy in the same market but with zero drift. The second strategy is the optimal strategy in the market with drift, but with a zero initial asset position. This second strategy in fact exploits the knowledge about the future drift to make a profit. It is however completely independent of the original liquidation problem; we therefore neglect it in this analysis of optimal liquidation and focus on the first strategy, which can be computed under the assumption of zero drift. Mathematically, the assumption of zero drift is necessary since we will consider liquidations over infinite time horizons, and only in the absence of drift can we expect an investor to actually liquidate a portfolio without a finite time constraint.[3]

[3] In the case of non-zero drift we can expect an investor to target a non-zero portfolio that will use the drift information to generate positive returns. While trading towards this target portfolio, in certain cases the investor may momentarily hold a zero portflio, but will subsequently continue trading towards the non-zero target portfolio.

In the following, we will not be concerned with P itself, but with the proceeds $\mathrm{P}^\top \xi$ of trading. Several different functions $\mathrm{TempImp} : \mathbb{R}^n \to \mathbb{R}^n$ have the same effect on $\mathrm{P}^\top \xi$. For example, in the two asset case the temporary impact functions $\mathrm{TempImp}(\xi)$ and

$$\widetilde{\mathrm{TempImp}}(\xi) := \mathrm{TempImp}(\xi) + \xi^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

give the same proceeds $\mathrm{P}^\top \xi$. We will therefore not specify TempImp, but instead will work directly with the "temporary impact cost of trading"

$$f : \xi \in \mathbb{R}^n \to f(\xi) := \mathrm{TempImp}(\xi)^\top \xi \in \mathbb{R}_0^+.$$

We assume that f is C^1 on $\mathbb{R}^n$, and that it is C^2 and larger than zero on $\mathbb{R}^n \backslash \{0\}$. Furthermore, we require that f has a positive-definite Hessian matrix $D^2 f$ on $\mathbb{R}^n \backslash \{0\}$, or equivalently that is has a non-singular Hessian matrix and is convex. Additionally, we need to assume that f scales like a power law in the trading speed ξ. More precisely, we assume that there is a constant $\alpha \in \mathbb{R}^+$ such that for all $r \in \mathbb{R}_0^+$:

$$f(r\xi) = r^{\alpha+1} f(\xi). \tag{1}$$

Note that this implies $f(0) = 0$. Irrespective of the choice of α, a higher trading speed will result in faster liquidation and an increased temporary impact cost of trading. E.g., if $\alpha = 1$, then doubling the trading speed will increase the temporary impact cost by a factor of 2 per unit traded. For a discussion of the relevance of the scaling property, see the remark after Theorem 5.2.

Several market models fit into our framework. The non-linear temporary impact models for a single asset discussed by Almgren (2003) and statistically estimated by Almgren, Thum, Hauptmann, and Li (2005) correspond to $f(\xi) = \lambda \xi^\beta$. For multiple assets, the linear model introduced by Almgren and Chriss (2001) and analysed by Konishi and Makimoto (2001) can be realized in our framework by setting $f(\xi) = \xi^\top \Lambda \xi$ with a matrix $\Lambda \in \mathbb{R}^{n \times n}$. A non-linear version of this model is given by $f(\xi) = (\xi^\top \Lambda \xi)^\beta$. Many other model choices fulfil our assumptions.[4]

We assume that ξ is progressively measurable with respect to a filtration in which B is a Brownian motion, that $\int_0^\infty f(\xi_t) dt < \infty$ and that each component of the resulting portfolio $\mathrm{X}_t^\xi(\omega)$ is bounded uniformly in t and ω with upper and lower bounds that may depend on the choice of ξ. Note that strategies are allowed to (at least temporarily) increase positions as well as to change the sign of positions (e.g., turn a long position in a given asset into a short position). Furthermore they are not required to reach a zero position at any point in time.

In the following we assume that the investor is a von-Neumann-Morgenstern investor with a utility function $u : \mathbb{R} \to \mathbb{R}$ with absolute risk aversion $A(M)$ that is bounded away from zero and infinity:

$$A(M) := -\frac{u_{MM}(M)}{u_M(M)}$$

$$0 < \inf_{M \in \mathbb{R}} A(M) =: A_{min} \leq \sup_{M \in \mathbb{R}} A(M) =: A_{max} < \infty$$

Furthermore, we assume that the utility function u is sufficiently smooth (C^6).[5]

3 Investor's optimal liquidation objective

We now define the problem of optimal liquidation in the illiquid market model. We consider a large investor who needs to sell a position of X_0 shares of a risky asset and already holds m units of cash. When following a

[4] Examples include $f((\xi_{(1)}, \xi_{(2)})^\top) = \xi_{(1)}^6 + \xi_{(1)}^4 \xi_{(2)}^2 + \xi_{(1)}^2 \xi_{(2)}^4 + \xi_{(2)}^6$ and $f((\xi_{(1)}, \xi_{(2)})^\top) = (\xi_{(1)}^2 + \xi_{(2)}^2) \exp(\xi_{(1)}^2 / (\xi_{(1)}^2 + \xi_{(2)}^2))$.

[5] We will use that u is C^4 (i.e., A is C^2) in several central definitions and statements. E.g., in Theorem 5.2, we rely on the existence of $\tilde{a}_{MM}$ where $\tilde{a}(0, M) = A(M)^{\frac{1}{\alpha+1}}$. We will use the stronger assumption of u being C^6 to show the smoothness of $\tilde{w}$ in Proposition A.10, which in turn is required for Equation 49 in Proposition A.11.

trading strategy ξ, the investor's total cash position is given by

$$\begin{aligned}\mathcal{M}_t(\xi) &= m + \int_0^t \mathrm{P}_s \xi_s \, ds \\ &= m + \tilde{\mathrm{P}}_0 \mathrm{X}_0 - \frac{1}{2}(\mathrm{X}_0)^\top \Gamma \mathrm{X}_0 + \underbrace{\int_0^t (\mathrm{X}_s^\xi)^\top \sigma \, d\mathrm{B}_s}_{\Phi_t} - \int_0^t f(\xi_s)\, ds \\ &\quad \underbrace{-\tilde{\mathrm{P}}_0 \mathrm{X}_t^\xi - \frac{1}{2}\Big((\mathrm{X}_t^\xi)^\top \Gamma \mathrm{X}_t^\xi - 2(\mathrm{X}_0)^\top \Gamma \mathrm{X}_t^\xi\Big) - (\mathrm{X}_t^\xi)^\top \sigma \mathrm{B}_t}_{\Psi_t}.\end{aligned}$$

We neglect the accumulation of interest.[6] It is not clear a priori that this is acceptable, since over long time horizons a positive interest rate could potentially have a significant impact on wealth dynamics. We will see that even without interest, the asset position decreases quickly under the optimal trading strategy. Incorporating a positive interest rate will lead to an even faster decrease of the asset position; however, due to the already fast exponential liquidation, only small changes to the optimal trading strategy are expected for reasonable parameters.

Since the large investor intends to sell the asset position, we expect the liquidation proceeds to converge $\mathbb{P}$-a.s. to a (possibly infinite) limit as $t \to \infty$. Convergence of Φ_t follows if

$$\mathbb{E}\Big[\int_0^\infty (\mathrm{X}_s^\xi)^\top \Sigma \mathrm{X}_s^\xi \, ds\Big] < \infty \tag{2}$$

and a.s. convergence of Ψ_t is guaranteed if a.s.

$$\lim_{t\to\infty} (\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi t \ln \ln t = 0. \tag{3}$$

We will regard strategies *admissible* if they satisfy the preceding two conditions in addition to the assumptions in Section 2. By $\mathcal{X}$ we denote the class of all admissible strategies ξ; for notational simplicity, we will not make the dependence on X_0 explicit.[7] Setting

$$M_t^\xi = \underbrace{m + \tilde{\mathrm{P}}_0 \mathrm{X}_0 - \frac{1}{2}(\mathrm{X}_0)^\top \Gamma \mathrm{X}_0}_{M_0} + \int_0^t (\mathrm{X}_s^\xi)^\top \sigma \, d\mathrm{B}_s - \int_0^t f(\xi_s)\, ds, \tag{4}$$

for $\xi \in \mathcal{X}$ we then have

$$M_\infty^\xi := \lim_{t\to\infty} \mathcal{M}_t(\xi). \tag{5}$$

All of the five terms adding up to M_t^ξ can be interpreted economically. The number m is simply the initial cash endowment of the investor. $\tilde{\mathrm{P}}_0 \mathrm{X}_0$ is the face value of the initial position. The term $\frac{1}{2}(\mathrm{X}_0)^\top \Gamma \mathrm{X}_0$ corresponds to the liquidation costs resulting from the permanent price impact of ξ. Due to the linearity of the permanent impact function, it is independent of the choice of the liquidation strategy. The stochastic integral corresponds to the volatility risk that is accumulated by selling throughout the interval $[0,\infty[$ rather than liquidating the portfolio instantaneously. The integral $\int_0^\infty f(\xi_s)\, ds$ corresponds to the transaction costs arising from temporary market impact.

We assume that the investor wants to maximize the expected utility of her cash position after liquidation:[8]

$$v(\mathrm{X}_0, M_0) := \sup_{\xi\in\mathcal{X}} \mathbb{E}[u(\mathcal{M}_\infty^\xi)] = \sup_{\xi\in\mathcal{X}} \mathbb{E}[u(M_\infty^\xi)]. \tag{6}$$

[6] Similarly we disregard any income or expenditure related to securities lending. If lending asset generates the same income as interest then these two effects can cancel out even if assumed to be non-zero.

[7] $\mathcal{X}$ contains adapted strategies that respond dynamically to changes in market prices. In Section 4, we will introduce the set $\bar{\mathcal{X}} \subset \mathcal{X}$ of deterministic admissible strategies.

[8] Alternatively, the maximisation of $\lim_{t\to\infty} \mathbb{E}[u(M_t^\xi)]$ can be considered, for which the constraints of Equations (2) and (3) can be dropped.

4 Deterministic strategies and mean-variance optimization

Before considering the dynamic maximization of expected utility, we start our analysis with deterministic mean-variance optimization. Let $\bar{\mathcal{X}} \subset \mathcal{X}$ be the set of deterministic admissible strategies. We consider the mean-variance value function[9]:

$$\bar{v}(\mathrm{X}_0) := \inf_{\bar{\xi} \in \bar{\mathcal{X}}} \left[\int_0^\infty f(\bar{\xi}_s) ds + \frac{1}{2} \int_0^\infty (\mathrm{X}_s^{\bar{\xi}})^\top \Sigma \mathrm{X}_s^{\bar{\xi}} ds \right]. \tag{7}$$

Mean-variance optimisation has frequently been studied in its own right. In the following section, we will use deterministic mean-variance optimal strategies to construct optimal dynamic strategies for utility maximisation and thus reveal an underlying connection between these seemingly different modelling approaches. The following theorem establishes the existence of an optimal trading strategy $\bar{\xi}$ and provides some of its features.

Theorem 4.1. *For each $\mathrm{X}_0 \in \mathbb{R}^n$, there is a unique minimizer $\bar{\xi}^{(\mathrm{X}_0)}$ of Equation (7). This minimizer satisfies Bellman's principle of optimality, i.e., there is a continuous vector field*

$$\bar{\mathrm{a}} : \mathrm{X} \in \mathbb{R}^n \to \bar{\mathrm{a}}(\mathrm{X}) \in \mathbb{R}^n$$

such that for all $\mathrm{X}_0 \in \mathbb{R}^n$ and each $t \in \mathbb{R}_0$, we have

$$\bar{\xi}_t^{(\mathrm{X}_0)} = \bar{\mathrm{a}} \left(\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}} \right).$$

Furthermore, the vector field $\bar{\mathrm{a}}$ fulfils the following two equations:

$$\nabla f(\bar{\mathrm{a}}(\mathrm{X})) = \bar{v}_{\mathrm{X}} \text{ for all } \mathrm{X} \in \mathbb{R}^n \tag{8}$$

$$\frac{f(\bar{\mathrm{a}}(\mathrm{X}))}{\mathrm{X}^\top \Sigma \mathrm{X}} = \frac{1}{2\alpha} \text{ for all } \mathrm{X} \in \mathbb{R}^n \backslash \{0\}. \tag{9}$$

Equation (8) is the dynamic programming PDE (see Cesari (1983)) and Equation (9) is the Beltrami identity (see Beltrami (1868) or Weisstein (2002)). The proof of Theorem 4.1 first investigates deterministic mean-variance liquidation strategies for a finite liquidation time horizon T; the results for the infinite time horizon problem (7) are subsequently derived by considering $T \to \infty$.

For special cases, the vector field $\bar{\mathrm{a}}$ and the mean-variance value function $\bar{v}$ are available in closed form. For the single asset case with non-linear temporary impact $f(\xi) = \lambda \xi^{\alpha+1}$, Almgren (2003) derived

$$\bar{a}(X) = \left(\frac{\sigma^2 X^2}{2\alpha\lambda} \right)^{\frac{1}{\alpha+1}}$$
$$\bar{v}(X) = \frac{(\alpha+1)^2}{3\alpha+1} \left(\frac{\lambda \sigma^{2\alpha} X^{3\alpha+1}}{(2\alpha)^\alpha} \right)^{\frac{1}{\alpha+1}}. \tag{10}$$

For the multiple asset case, it is harder to find a closed form expression for $\bar{\mathrm{a}}$. However, if the temporary impact is linear, i.e., $f(\xi) = \xi^\top \Lambda \xi$, Λ is a diagonal matrix and $\Lambda^{-1}\Sigma$ has n different positive eigenvalues, then it is easy to derive from the formulas in Konishi and Makimoto (2001) that

$$\bar{\mathrm{a}}(\mathrm{X}) = \frac{1}{\sqrt{2}} \sqrt{\Lambda^{-1}\Sigma} \mathrm{X} \tag{11}$$

$$\bar{v}(\mathrm{X}) = \frac{1}{\sqrt{2}} \mathrm{X}^\top \Sigma \sqrt{\Sigma^{-1}\Lambda} \mathrm{X}. \tag{12}$$

It is straightforward to verify that the closed form expressions of Equations (10), (11) and (12) are consistent with Theorem 4.1.

Figures 1 and 2 illustrate the trajectories of the optimal deterministic trading strategies in the case of two positively correlated assets with covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

[9] More precisely, the function $\bar{v}$ is a simple transformation of the mean-variance value function.

In Figure 1, the trajectories for different portfolios X_0 are compared for fixed

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The correlation of the two assets connects the trading in both assets by the investor's desire to reduce portfolio risk by hedging. If the initial asset position in one of the assets is zero, it will not remain zero during the portfolio liquidation; instead, a long or short position is acquired that serves as a hedge for the initial non-zero position in the other asset. For the same reason, a portfolio with long positions in both assets might have a short position in one of the two assets during the optimal liquidation. This short position again serves as a hedge for the long position in the other asset; under certain conditions, it is cheaper to reduce risk by building up the short position as a hedge instead of by selling the long position quicker. For two example portfolios

$$\mathrm{X}_0 = \begin{pmatrix} \pm 1 \\ 1.5 \end{pmatrix},$$

the trajectories for different temporary impact matrices

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & d_2 \end{pmatrix}$$

with $d_2 \in [e^{-3}, e^3]$ are shown in Figure 2. The larger the differences in liquidity of the two assets, the larger the incentive to hedge the market risk by trading the more liquid asset quicker than the less liquid asset. For the portfolio

$$\mathrm{X}_0 = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix},$$

this effect is strong, since the initial portfolio market risk is high; for the portfolio

$$\mathrm{X}_0 = \begin{pmatrix} -1 \\ 1.5 \end{pmatrix},$$

the market risk is low already at the beginning of trading and thus the optimal trading trajectories are similar for different temporary impact matrices Λ.[10]

As can be seen from these pictures, in some cases it can be optimal to temporarily increase the initial position in one of the assets. In other cases, it is best to turn an initial long position into a short position (or vice versa). Furthermore the optimal strategies never complete liquidation in a finite time horizon; instead they converge towards a zero portfolio over time (without ever reaching it).

5 Dynamic maximization of expected utility

We now turn to the *dynamic* maximization of expected utility.

Theorem 5.1. *The value function v is a classical solution of the Hamilton-Jacobi-Bellman equation*

$$\inf_{\mathrm{a}} \left[-\frac{1}{2} v_{MM} \mathrm{X}^\top \Sigma \mathrm{X} + v_M f(\mathrm{a}) + v_{\mathrm{X}} \mathrm{a} \right] = 0 \tag{13}$$

with boundary condition

$$v(0, M) = u(M) \text{ for all } M \in \mathbb{R}. \tag{14}$$

The a.s. unique optimal control $\hat{\xi}_t$ is Markovian. We write it in feedback form as

$$\hat{\xi}_t = \mathrm{a}(\mathrm{X}_t^{\hat{\xi}}, M_t^{\hat{\xi}}). \tag{15}$$

For the value function, we have convergence:

$$v(\mathrm{X}_0, M_0) = \lim_{t \to \infty} \mathbb{E}[u(M_t^{\hat{\xi}})] = \mathbb{E}[u(M_\infty^{\hat{\xi}})].$$

[10] For the specific choice of price impact cost function f made in Figures 1 and 2, the liquidation direction converges to the last eigenvector of $\sqrt{\Lambda^{-1}\Sigma}$. In Figure 1, this is the vector $(-1, 1)$. In Figure 2, the vector depends on d_2: For $d_2 = 1$ we obtain the same asymptote $(-1, 1)$, for $d_2 \to \infty$ it is $(-0.5, 1)$ and for $d_2 \to 0$ it is $(1, -0.5)$. Note that the execution trajectories for different values of d_2 can intersect each other.

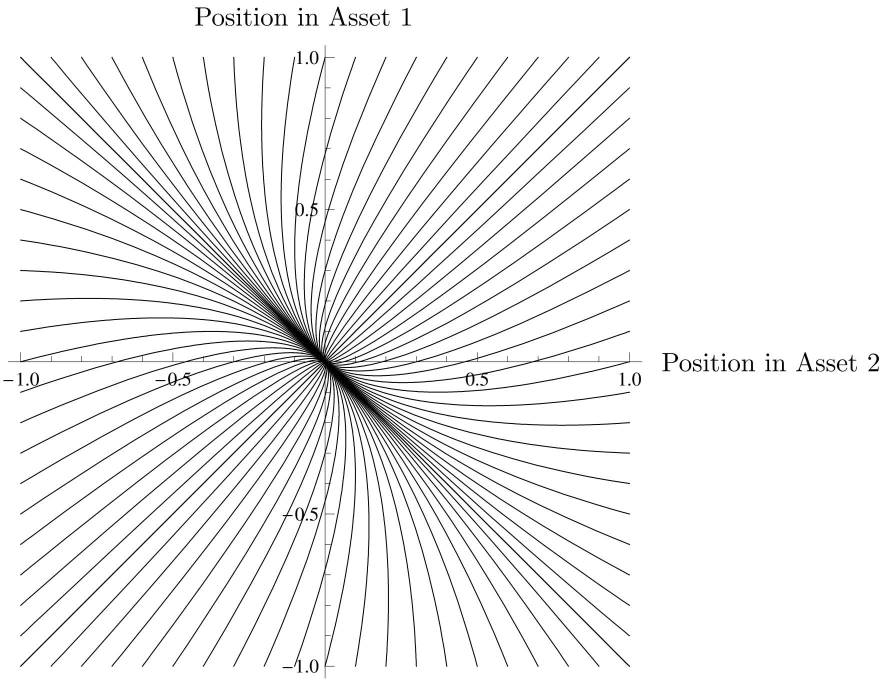

Figure 1: Parametric plot of the portfolio trajectories $\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}}$ under the mean-variance optimal deterministic strategy for different initial portfolios X_0. $\Lambda = ((1,0),(0,1))$, $\Sigma = ((1,0.5),(0.5,1))$.

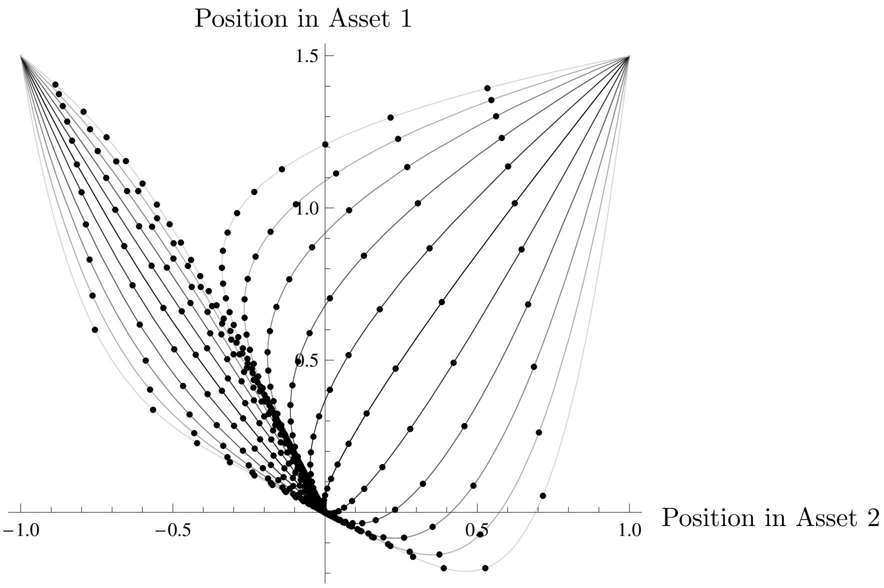

Figure 2: Parametric plot of the portfolio trajectories $\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}}$ under the mean-variance optimal deterministic strategy for two different initial portfolios X_0 and different temporary impact matrices $\Lambda = ((1,0),(0,d_2))$. Darker lines correspond to d_2 closer to 1. $\Sigma = ((1,0.5),(0.5,1))$. The dots show the portfolio X_{t_i} at time points $t_i = i/2$ for $i \in \mathbb{N}$.

Note that in Equation (13) and in the rest of this chapter, we use the shorthand notation $v_{\mathrm{X}} = \nabla_{\mathrm{X}} v$. Once we have a candidate value function (and optimal strategy) satisfying Equation 13, the proof of Theorem 5.1 is a (non-standard) verification argument. But the existence of a solution to the HJB Equation (13) is far from obvious; even for the simple integration of vector fields in $\mathbb{R}^n \times \{0\}$, integration conditions need to be fulfilled. Fortunately, the construction of the optimal control a and the value function v can be reduced to a two-dimensional problem involving only the portfolio value M and the mean-variance liquidation cost $Y = \bar{v}(\mathrm{X})$.

Theorem 5.2. *The optimal control* a *is given by*

$$\mathrm{a}(\mathrm{X}, M) = \tilde{a}(\bar{v}(\mathrm{X}), M)\bar{\mathrm{a}}(\mathrm{X})$$

with a "relative liquidation speed" function $\tilde{a} : (Y, M) \in \mathbb{R}_0^+ \times \mathbb{R} \to \tilde{a}(Y, M) \in \mathbb{R}^+$ *that is the unique classical solution of the fully non-linear parabolic PDE*

$$\tilde{a}_Y = -\frac{2\alpha+1}{\alpha+1}\tilde{a}^\alpha \tilde{a}_M + \frac{\alpha(\alpha-1)}{\alpha+1}\left(\frac{\tilde{a}_M}{\tilde{a}}\right)^2 + \frac{\alpha}{\alpha+1}\frac{\tilde{a}_{MM}}{\tilde{a}} \tag{16}$$

with initial condition

$$\tilde{a}(0, M) = A(M)^{\frac{1}{\alpha+1}}. \tag{17}$$

The bounds of the absolute risk aversion determine bounds of the relative liquidation speed $\tilde{a}$*:*

$$\inf_{(Y,M)\in\mathbb{R}_0^+\times\mathbb{R}} \tilde{a}(Y, M) = \inf_{M\in\mathbb{R}} \tilde{a}(0, M) =: \tilde{a}_{min} = (A_{min})^{\frac{1}{\alpha+1}}$$
$$\sup_{(Y,M)\in\mathbb{R}_0^+\times\mathbb{R}} \tilde{a}(Y, M) = \sup_{M\in\mathbb{R}} \tilde{a}(0, M) =: \tilde{a}_{max} = (A_{max})^{\frac{1}{\alpha+1}}.$$

For $\alpha = 1$, Equations (16) and (17) describing the relative liquidation speed $\tilde{a}$ are a special case of Equations (21) and (22) for the transformed optimal control in Schied and Schöneborn (2009) with $\lambda = 1$ and $\sigma^2 = 2$.[11]

While the "Separation Theorem" 5.2 might feel surprising, it is possible to interpret it intuitively in light of previous results on optimal liquidation. First, Schied and Schöneborn (2009) show that general utility maximizing investors liquidate single asset positions similar to CARA investors with a risk aversion level that is a weighted average of the risk aversion at possible outcomes of the liquidation value. Second, Schied, Schöneborn, and Tehranchi (2010) show that investors with a CARA utility function in turn behave like mean variance investors when liquidating a basket portfolio over a finite time horizon in a market with linear price impact. Theorem 5.2 confirms that both of these connections also hold for basket liquidation over an infinite time horizon with general price impact and general utility functions.

The necessity of the scaling property (Equation (1)) can be understood when considering the simplest case of power law price impact for uncorrelated assets with no cross-asset price impact. If the power law exponent for the price impact of the ith asset is α_i, then the existing literature suggests that the optimal liquidation speed for the ith asset scales as $A^{\frac{1}{\alpha_i+1}}$ in the level of risk aversion A. For the Separation Theorem to hold, we need the ratio of the liquidation speeds in any two assets i and j to be independent of the level of risk aversion A, i.e., we need $\alpha_i = \alpha_j$, which in turn is equivalent to the scaling property. Theorem 5.2 establishes that the scaling property is not only necessary but also sufficient.

Because of Theorem 5.2, utility maximization becomes numerically achievable for practical applications. Bertsimas, Hummel, and Lo (1999) find that even the minimization of expected liquidation costs is numerically challenging for large portfolios. While mean-variance optimal liquidation is by now a standard service of many banks, a utility maximizing dynamic liquidation by brute-force methods of dynamic programming appears out of reach. By Theorem 5.2, such a brute-force approach is fortunately not necessary.

Theorem 5.3. *The value function is given by*

$$v(\mathrm{X}, M) = \tilde{v}(\bar{v}(\mathrm{X}), M)$$

[11] Note that here the relative liquidation speed $\tilde{a}$ describes the length of the utility-maximizing control a with respect to the length of the mean-variance optimal control $\bar{\mathrm{a}}$, while the transformed optimal control $\tilde{c}$ in Schied and Schöneborn (2009) described the magnitude of c with respect to the portfolio size X. For $\alpha = 1$ as in the linear model of Schied and Schöneborn (2009), portfolio size X and mean-variance optimal trading speed $\bar{a}$ are proportional; this is not necessarily the case for $\alpha \neq 1$.

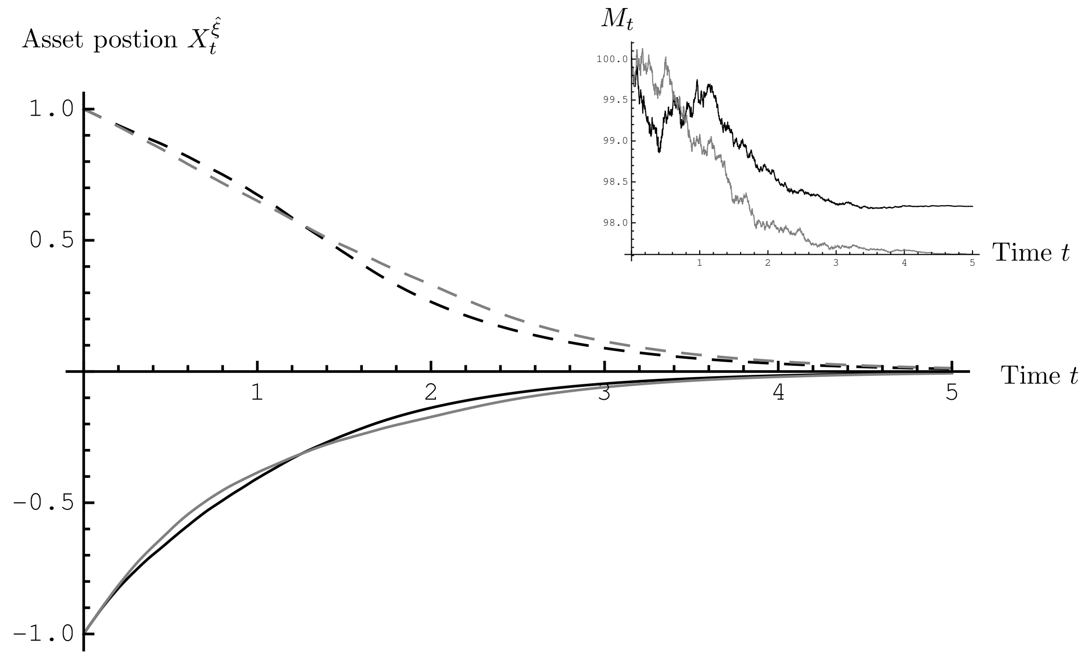

Figure 3: Two sample optimal execution paths $\mathrm{X}_t^{\hat{\xi}}$ corresponding to different paths of the Brownian motion B_t. The inset shows the corresponding evolution of the variable M_t. Black lines represent the first scenario and grey lines the second; solid lines represent the first asset while dashed lines represent the second asset. Parameters are $\Sigma = ((1, 0.5), (0.5, 1))$, $\Lambda = ((0.3, 0), (0, 6))$, $X_0 = (-1, 1)$, $P_0 = (100, 100)$, $M_0 = 100$ and the utility function with absolute risk aversion $A(M) = 2.4 + 2\tanh(10(M - 97))$. 2000 simulation steps were used covering the time span $[0, 5]$.

with a function $\tilde{v} : (Y, M) \in \mathbb{R}_0^+ \times \mathbb{R} \to \tilde{v}(Y, M) \in \mathbb{R}$ *that is the unique classical solution of the non-linear first order PDE*

$$\tilde{v}_Y = -\tilde{v}_M \tilde{a}^\alpha \tag{18}$$

with initial condition

$$\tilde{v}(0, M) = u(M). \tag{19}$$

Theorems 5.2 and 5.3 reveal a tight connection between mean-variance optimization and maximization of expected utility. Both approaches lead to the same liquidation strategy, they only differ by the speed with which this strategy is executed. The expected utility of optimal liquidation then depends only on the current portfolio value M and the mean-variance costs of deterministic liquidation $\bar{v}(\mathrm{X})$. The proof of Theorems 5.1, 5.2 and 5.3 is achieved jointly. First, solutions to the PDEs (16) and (18) are established and shown to yield a value function satisfying the HJB equation (13). Subsequently a verification theorem establishes that this solution to the HJB equation is indeed the value function.

Corollary 5.4. *The asset position* $\mathrm{X}_t^{\hat{\xi}}$ *at time t under the optimal control* $\hat{\xi}$*, is given by*

$$\mathrm{X}_t^{\hat{\xi}} = \mathrm{X}^{\bar{\xi}}_{\int_0^t \tilde{a}(\bar{v}(\mathrm{X}_s^{\hat{\xi}}), M_s^{\hat{\xi}})\, ds}. \tag{20}$$

This corollary follows since both $\mathrm{X}_t^{\hat{\xi}}$ and $\mathrm{X}^{\bar{\xi}}_{\int_0^t \tilde{a}(\bar{v}(\mathrm{X}_s^{\hat{\xi}}), M_s^{\hat{\xi}})\, ds}$ satisfy $\frac{d}{dt}\mathrm{X}_t = -\tilde{a}(\bar{v}(\mathrm{X}_t), M_t)\bar{\mathrm{a}}(\mathrm{X}_t)$. Figure 3 illustrates how liquidations can evolve depending on the utility function and asset price evolutions. The two examples shown reflect different asset price scenarios. Both examples follow the same portfolio execution strategy, but executed at different speeds dynamically driven by the random changes in portfolio value. Initially the positions in both assets get liquidated slower in the first scenario (black lines in the figure) compared to the second scenario (grey lines), but after $t \sim 0.8$ the execution in the first scenario speeds up. Since the same execution strategy is being followed in both scenarios, this results in the execution of the first scenario catching up with the second in both assets at the same time $t \sim 1.3$.

It is now clear that the features of mean-variance optimal strategies are also exhibited by utility maximising strategies (e.g. increases in positions, change of sign of positions and gradual convergence to a zero portfolio). For investors with CARA utility, this connection is particularly strong.

Corollary 5.5. *For investors with a utility function $u(M) = -e^{-AM}$ with constant absolute risk aversion $A(M) \equiv A$, the optimal adaptive liquidation strategy is deterministic and is given by*

$$\mathrm{a}(\mathrm{X}, M) = A^{\frac{1}{\alpha+1}}\bar{\mathrm{a}}(\mathrm{X}) \tag{21}$$

$$\mathrm{X}_t^{\hat{\xi}} = \mathrm{X}^{\bar{\xi}}_{A^{\frac{1}{\alpha+1}}t} \tag{22}$$

$$v(\mathrm{X}, M) = -\exp\left(-AM + A^{\frac{2\alpha+1}{\alpha+1}}\bar{v}(\mathrm{X})\right) \tag{23}$$

This corollary follows since the strategy and value function proposed in (21) and (23) are the unique solutions of the PDEs (16) and (18) with their corresponding boundary conditions. For the example discussed in Section 3 ($f(\xi) = \xi^\top \Lambda \xi$, Λ a diagonal matrix, $\Lambda^{-1}\Sigma$ having n different positive eigenvalues), using Equations (11) and (12) we obtain for investors with CARA utility $u(M) = e^{-AM}$ that[12]

$$\mathrm{a}(\mathrm{X}, M) = A^{\frac{1}{\alpha+1}}\frac{1}{\sqrt{2}}\sqrt{\Lambda^{-1}\Sigma}\mathrm{X} \tag{24}$$

$$\mathrm{X}_t^{\hat{\xi}} = \exp\left(-A^{\frac{1}{\alpha+1}}\frac{1}{\sqrt{2}}\sqrt{\Lambda^{-1}\Sigma}t\right)\mathrm{X}_0 \tag{25}$$

$$v(\mathrm{X}, M) = -\exp\left(-AM + A^{\frac{2\alpha+1}{\alpha+1}}\frac{1}{\sqrt{2}}\mathrm{X}^\top\Sigma\sqrt{\Sigma^{-1}\Lambda}\mathrm{X}\right). \tag{26}$$

Since for $\alpha = 1$ Equations (16) and (17) are a special case of Equations (21) and (22) in Schied and Schöneborn (2009), all the results of Schied and Schöneborn (2009) that follow from the properties of Equation (21) carry over to the multiple asset setting when $\alpha = 1$. Several of them also hold for general α. The following two propositions exemplify this.

Theorem 5.6. $||\mathrm{a}(X, M)||$ *is increasing (decreasing) in M for all values of X if and only if the absolute risk aversion $A(M)$ is increasing (decreasing) in M.*

If the value of the portfolio rises, then M rises. A strategy with an optimal control a that is increasing in M (everything else held constant) sells fast in such a scenario, i.e., is aggressive in-the-money; if a is decreasing in M, it is passive in-the-money, and if a is independent of M, then the strategy is neutral in-the-money. It follows from the preceding theorem that $A(M)$ determines this characteristic of the optimal strategy:

Utility function		Optimal trading strategy
Decreasing absolute risk aversion (DARA)	$\Leftrightarrow$	Passive in-the-money (PIM)
Constant absolute risk aversion (CARA)	$\Leftrightarrow$	Neutral in-the-money (NIM)
Increasing absolute risk aversion (IARA)	$\Leftrightarrow$	Aggressive in-the-money (AIM)

Figure 3 shows example liquidations for a utility function with increasing absolute risk aversion (IARA). As expected, the execution speeds up whenever M is rising and slows down whenever M is falling.

The proof of Theorem 5.6 crucially relies on the following theorem.

Theorem 5.7. *Suppose u^0 and u^1 are two utility functions such that u^1 has a higher absolute risk aversion than u^0, i.e., $A^1(M) \geq A^0(M)$ for all M. Then an investor with utility function u^1 liquidates the same portfolio X_0 faster than an investor with utility function u^0. More precisely, the corresponding optimal strategies satisfy*

$$\tilde{a}^1 \geq \tilde{a}^0. \tag{27}$$

The proof of Theorem 5.7 relies on a Feynman-Kac argument.

The "Separation Theorem" 5.2 does not hold for basket liquidations with a finite time horizon T. Let us consider a simple example of two uncorrelated assets with the same volatility but different liquidity:

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}.$$

[12] Schied, Schöneborn, and Tehranchi (2010) derive the optimal liquidation strategy and value function for a CARA investor in the single asset case but with a finite liquidation time horizon $T < \infty$. As $T \to \infty$, their results converge to the expressions in Equations 24 to 26.

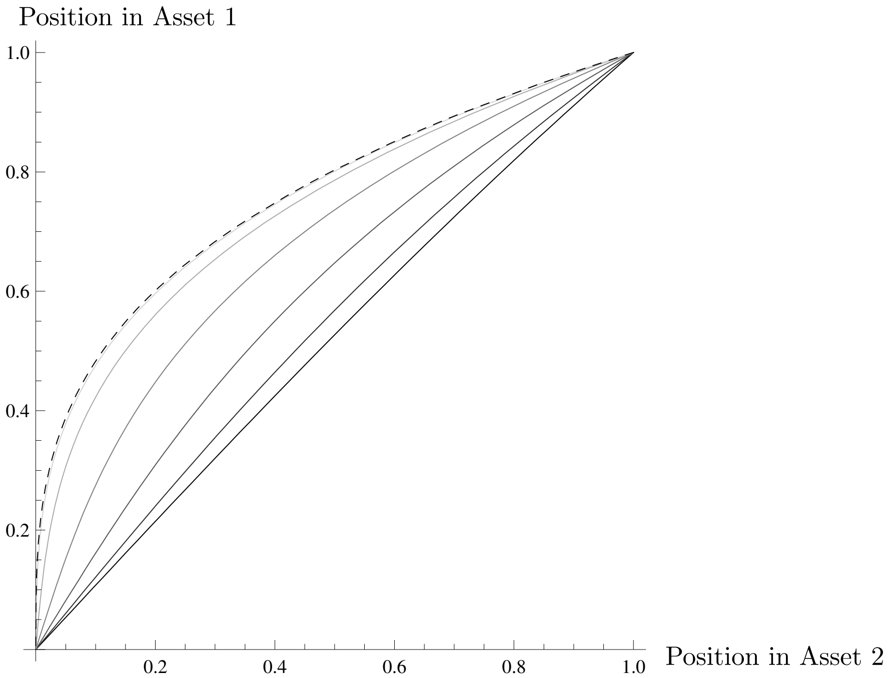

Figure 4: Parametric plot of the optimal trading trajectories for a finite liquidation time horizon T for CARA investors with different levels of absolute risk aversion $A \in [1, e^5]$ (darker lines correspond to lower risk aversion). The dashed black line is the trajectory of the optimal liquidation strategy with an infinite time horizon. $\mathrm{X}_0 = (1,1)^\top$, $\Lambda = ((1,0),(0,10))$, $\Sigma = ((1,0),(0,1))$.

By the results of Schied, Schöneborn, and Tehranchi (2010), the optimal strategy for CARA investors is the optimal deterministic strategy for mean-variance investors. For mean-variance investors however there is no interaction between the liquidation of the positions in the two assets due to their independence. Hence the optimal strategy liquidates both asset positions independently with the strategy given in Theorem 2.3 in Schied, Schöneborn, and Tehranchi (2010). Investors with an infinite time horizon have an incentive to move their portfolio in the direction of lower risk; if they have higher risk aversion then they will do this quicker than if they have lower risk aversion, but the trading direction is unchanged. If a finite time liquidation horizon is imposed, then the investor has an additional incentive to move their portfolio towards zero to satisfy the portfolio liquidation constraint; this incentive is independent of the investor's risk aversion. The optimal trading strategy for an investor with finite time horizon is blending these two incentives, but the relative weight given to the two depends on the investor's level of risk aversion. A higher level of risk aversion results in trading predominantly in a risk reducing direction, while a lower level of risk aversion favours trading towards an empty portfolio. Figure 4 illustrates how the trajectory of the optimal liquidation strategy depends on the level of absolute risk aversion A of the utility function. For large values of risk aversion, the basket is liquidated quickly irrespective of the time horizon. Imposing a finite liquidation horizon hence results in only a small change to the liquidation trajectory, which is primarily driven by market liquidity and volatility. For small values of the risk aversion A however the infinite time horizon liquidation strategy has only liquidated a small proportion of the portfolio by time T. Imposing complete liquidation by T hence requires a significant change of the liquidation strategy. The optimal strategy corresponds roughly to a linear reduction in asset position towards an empty portfolio since the primary trading motivation is the time constraint.

A Proof of results

This appendix consists of three parts. First we discuss mean-variance optimal strategies and prove Theorem 4.1. By extending methods of calculus of variations to the infinite time setting, we show that optimal strategies exist, that they are unique and that they satisfy Bellman's principle of optimality. In the second subsection, we show that a smooth solution of the HJB equation exists and provide some of its properties. This is achieved by first obtaining a solution of the PDE for $\tilde{a}$ and then defining $\tilde{v}$ by a transport equation with coefficient $\tilde{a}$. In the third subsection, we apply a verification argument and show that this solution of the HJB equation must be equal to the value function. Theorems 5.1, 5.2 and 5.3 and Corollary 5.4 are direct consequences of the propositions in the last two subsections. The proofs in the last two subsections have a similar structure to the proofs in Schied and Schöneborn (2009). However, they differ in a few subtle points and we therefore provide them in full detail.

A.1 Optimal mean-variance strategies

To obtain optimal trading strategies for the infinite horizon setting, we will first show that optimal strategies exist for the setting with finite horizon T (i.e., $\mathrm{X}_t = 0$ for $t \geq T$) and then consider the limit $T \to \infty$.

Lemma A.1. *If a mean-variance optimal trading strategy exists for* $\mathrm{X}_0 \in \mathbb{R}^n$ *and time horizon* $T \in]0,\infty]$*, then this strategy is unique.*

Proof. This follows directly from the strict convexity of the functional $f(\xi) + \frac{1}{2}\mathrm{X}^\top \Sigma \mathrm{X}$. □

Proposition A.2. *For* finite *liquidation time horizons* $T \in \mathbb{R}^+$*, a mean-variance optimal liquidation strategy* $\xi^{(\mathrm{X}_0,T)}$ *exists for all initial portfolios* $\mathrm{X}_0 \in \mathbb{R}^n$*. The portfolio evolution* $\mathrm{X}_t^{\xi^{(\mathrm{X}_0,T)}}$ *is* C^1 *in* t *(i.e., the optimal trading vector* $\xi_t^{(\mathrm{X}_0,T)}$ *is continuous). We denote the time at which the portfolio* X_t^ξ *attains zero by*

$$T_0 := \inf\{t > 0 : \mathrm{X}_t^\xi = 0\} \in]0, T].$$

For $t \in [0, T_0[$*, the portfolio evolution* X_t^ξ *is even* C^2 *and fulfils the Euler-Lagrange equation*

$$\Sigma \mathrm{X}_t = D^2 f(-\dot{\mathrm{X}}_t)\ddot{\mathrm{X}}_t.$$

The optimal trading vector $\xi^{(\mathrm{X}_0,T)}$ *satisfies Bellman's principle of optimality, i.e.,*

$$\xi_t^{(\mathrm{X}_0,T)} = \xi_0^{(\mathrm{X}_t,T-t)}.$$

Furthermore, the initial trading speed ξ_0 *is locally uniformly bounded. More precisely, for each portfolio* $\bar{\mathrm{X}}_0 \in \mathbb{R}$ *and each time horizon* $\bar{T}$*, there is a* $\delta > 0$ *and* $C > 0$ *such that* $|\xi_0^{(\mathrm{X}_0,T)}| < C$ *for all* $|\mathrm{X}_0 - \bar{\mathrm{X}}_0| < \delta$ *and* $T \geq \bar{T}$*.*

Theorem 2.2 in Schied, Schöneborn, and Tehranchi (2010) establishes the existence of a mean-variance optimal strategy for finite liquidation time horizons, but not the uniform bound on ξ_0, which we need for our proof of Proposition A.3. We therefore present a self-contained proof of Proposition A.2 establishing this bound.

Proof. First, we observe that for mean-variance optimal ξ there is an a priori upper bound $K > 0$ independent of T such that

$$\sup\{|\mathrm{X}_t^\xi| : t \in [0,T]\} < K.$$

To see this, select an arbitrary $\tilde{K} > \mathrm{X}_0^\top \Sigma \mathrm{X}_0$ and assume that $\frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t$ attains $\tilde{K}$ at $T_2 := \min\{t > 0 : \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t \geq \tilde{K}\}$. Then

$$\frac{\tilde{K}}{2} \leq \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t \leq \tilde{K}$$

for all $t \in [T_1, T_2]$ with $T_1 := \max\{t < T_2 : \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t \leq \frac{\tilde{K}}{2}\}$. We therefore have

$$\int_{T_1}^{T_2} \left(f(\xi_t) + \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t \right) dt \geq \left(\min_{\substack{\tilde{\mathrm{X}} \in \mathscr{X} \text{ s.t.} \\ \frac{1}{2}\tilde{\mathrm{X}}_{T_1}^\top \Sigma \tilde{\mathrm{X}}_{T_1} = \frac{\tilde{K}}{2}, \frac{1}{2}\tilde{\mathrm{X}}_{T_2}^\top \Sigma \tilde{\mathrm{X}}_{T_2} = \tilde{K}}} \int_{T_1}^{T_2} f(\tilde{\xi}_t) dt \right) + (T_2 - T_1)\frac{\tilde{K}}{2}.$$

Let $\tilde{\mathrm{X}}^*$ respectively $\tilde{\xi}^*$ be a minimizer of the first term on the right hand side. Then $\frac{2}{\tilde{K}}\tilde{\mathrm{X}}^*_{T_1+t(T_2-T_1)}$ has derivative $\frac{2(T_2-T_1)}{\tilde{K}}\tilde{\xi}^*_{T_1+t(T_2-T_1)}$ and satisfies $\frac{1}{2}\tilde{\mathrm{X}}_0^\top \Sigma \tilde{\mathrm{X}}_0 = 1$ and $\frac{1}{2}\tilde{\mathrm{X}}_1^\top \Sigma \tilde{\mathrm{X}}_1 = 2$. Due to the scaling property

(Equation (1)), we have

$$\begin{aligned}\min_{\substack{\tilde{\mathrm{X}}\in\bar{\mathcal{X}}\text{ s.t.}\\ \frac{1}{2}\tilde{\mathrm{X}}_{T_1}^\top\Sigma\tilde{\mathrm{X}}_{T_1}=\frac{\tilde{K}}{2},\frac{1}{2}\tilde{\mathrm{X}}_{T_2}^\top\Sigma\tilde{\mathrm{X}}_{T_2}=\tilde{K}}}\int_{T_1}^{T_2} f(\tilde{\xi}_t)dt &= \int_{T_1}^{T_2} f(\tilde{\xi}_t^*)dt\\
&= (T_2-T_1)\int_0^1 f(\tilde{\xi}^*_{T_1+t(T_2-T_1)})dt\\
&= (T_2-T_1)\int_0^1 \left(\frac{\tilde{K}}{2(T_2-T_1)}\right)^{\alpha+1} f\left(\frac{2(T_2-T_1)}{\tilde{K}}\tilde{\xi}^*_{T_1+t(T_2-T_1)}\right)dt\\
&\geq \left(\frac{1}{T_2-T_1}\right)^{\alpha}\left(\frac{\tilde{K}}{2}\right)^{\alpha+1}\min_{\tilde{\mathrm{X}}\in\bar{\mathcal{X}}\text{ s.t. }\frac{1}{2}\tilde{\mathrm{X}}_0^\top\Sigma\tilde{\mathrm{X}}_0=1,\frac{1}{2}\tilde{\mathrm{X}}_1^\top\Sigma\tilde{\mathrm{X}}_1=2}\int_0^1 f(\tilde{\xi}_t)dt.\end{aligned}$$

Combining the previous two derivations, we obtain

$$\int_{T_1}^{T_2}\left(f(\xi_t)+\frac{1}{2}\mathrm{X}_t^\top\Sigma\mathrm{X}_t\right)dt \geq \left(\frac{1}{T_2-T_1}\right)^{\alpha}\left(\frac{\tilde{K}}{2}\right)^{\alpha+1}\tilde{C}+(T_2-T_1)\frac{\tilde{K}}{2} \tag{28}$$

with the constant

$$\tilde{C} := \min_{\tilde{\mathrm{X}}\in\bar{\mathcal{X}}\text{ s.t. }\frac{1}{2}\tilde{\mathrm{X}}_0^\top\Sigma\tilde{\mathrm{X}}_0=1,\frac{1}{2}\tilde{\mathrm{X}}_1^\top\Sigma\tilde{\mathrm{X}}_1=2}\int_0^1 f(\tilde{\xi}_t)dt > 0.$$

Since $\tilde{C}$ is independent of T_1 and T_2, the right-hand side of Equation (28) is bounded from below by a function of $\tilde{K}$ that is increasing and unbounded. This establishes that an optimal ξ cannot attain arbitrarily large values of $\mathrm{X}_t^\top\Sigma\mathrm{X}_t$ respectively $\sup_t|\mathrm{X}_t|$.

We can therefore reduce the optimization problem with unbounded $\mathrm{X}_t\in\mathbb{R}^n$ to an optimization problem with bounded $\mathrm{X}_t\in[-K,K]^n$. By Tonelli's existence theorem (see, e.g., Cesari (1983), Theorem 2.20), a mean-variance optimal trading strategy exist for the bounded optimization problem; by our previous considerations, this strategy is also optimal for the unbounded optimization problem $\mathrm{X}_t\in\mathbb{R}^n$, and we denote this strategy by $\xi^{(\mathrm{X}_0,T)}$.

In order to apply theorems ensuring continuity of even smoothness of ξ, we need to show that the optimal $\xi=\xi^{(\mathrm{X}_0,T)}$ is essentially bounded. The idea of the following proof is that if ξ trades extremely quickly at some points in time, then the mean-variance costs of ξ can be reduced by "smoothing" the trading speed, i.e., slowing down trading when it is fast and accelerating it when it is slow. To formalize this argument, we first observe that there are bounds $(\mathrm{X}_t^\xi)^\top\Sigma\mathrm{X}_t^\xi<K_0$ and $\int_0^T f(\xi_t)dt=K_1<\infty$, and we define

$$\mu : t\in\mathbb{R}\to\mu_t := \int_0^t \mathbb{1}_{f(\xi_s)\geq K_2}ds\in\mathbb{R},$$

where $K_2>0$ is a large, arbitrary constant. ξ is essentially bounded if there is a $K_2>0$ with $\mu\equiv 0$. We assume that $\mu\neq 0$ for all $K_2\in\mathbb{R}^+$ and establish a contradiction. We define the time transformation

$$\tilde{t}(t,s) := s\mu_t + \frac{T-s\mu_T}{T-\mu_T}(t-\mu_t).$$

For $0<s<\frac{T}{\mu_T}$, this transformation is a bijection $\tilde{t}(\cdot,s):[0,T]\to[0,T]$ satisfying $\tilde{t}(0,s)=0$ and $\tilde{t}(T,s)=T$. When using the variables $\tilde{t}$ and t in the following, we will always assume that they are connected by this bijection, i.e., that $\tilde{t}=\tilde{t}(t,s)$. We can now define a new portfolio evolution Y depending on s:

$$\mathrm{Y}^{(s)} : \tilde{t}\in\mathbb{R}_0^+\to\mathrm{Y}_{\tilde{t}}^{(s)} := \mathrm{X}_t.$$

The portfolio evolution $\mathrm{Y}^{(s)}$ is absolutely continuous and fulfils

$$\xi_{\tilde{t}}^{(s)} := -\frac{d}{d\tilde{t}}\mathrm{Y}_{\tilde{t}}^{(s)} = \begin{cases}\frac{1}{s}\xi_t & \text{for } f(\xi_t)\geq K_2\\ \frac{T-\mu_T}{T-s\mu_T}\xi_t & \text{for } f(\xi_t)<K_2.\end{cases}$$

Note that $\xi^{(1)} = \xi$. The mean-variance costs of executing $Y^{(s)}$ are given by

$$
\begin{aligned}
&\int_0^T (f(\xi_{\tilde t}^{(s)}) + (Y_{\tilde t}^{(s)})^\top \Sigma Y_{\tilde t}^{(s)}) d\tilde t \\
= \;& \int_{f(\xi_t)\geq K_2} (f(\xi_{\tilde t}^{(s)}) + (Y_{\tilde t}^{(s)})^\top \Sigma Y_{\tilde t}^{(s)}) d\tilde t \\
&+ \int_{f(\xi_t)< K_2} (f(\xi_{\tilde t}^{(s)}) + (Y_{\tilde t}^{(s)})^\top \Sigma Y_{\tilde t}^{(s)}) d\tilde t \\
= \;& s \int_{f(\xi_t)\geq K_2} \left(f\left(\frac{1}{s}\xi_t\right) + (X_t^\xi)^\top \Sigma X_t^\xi \right) dt \\
&+ \frac{T - s\mu_T}{T-\mu_T} \int_{f(\xi_t)< K_2} \left(f\left(\frac{T-\mu_T}{T - s\mu_T}\xi_t\right) + (X_t^\xi)^\top \Sigma X_t^\xi \right) dt \\
= \;& \left(\frac{1}{s}\right)^\alpha \int_{f(\xi_t)\geq K_2} f(\xi_t) dt + s \int_{f(\xi_t)\geq K_2} (X_t^\xi)^\top \Sigma X_t^\xi dt \\
&+ \left(\frac{T-\mu_T}{T - s\mu_T}\right)^\alpha \int_{f(\xi_t)< K_2} f(\xi_t) dt + \frac{T - s\mu_T}{T-\mu_T} \int_{f(\xi_t)< K_2} (X_t^\xi)^\top \Sigma X_t^\xi dt.
\end{aligned}
$$

By differentiating with respect to s at $s = 1$, we have

$$
\begin{aligned}
&\left.\frac{d}{ds}\right|_{s=1} \int_0^T (f(\xi_{\tilde t}^{(s)}) + (Y_{\tilde t}^{(s)})^\top \Sigma Y_{\tilde t}^{(s)}) d\tilde t \\
&\quad = -\alpha \underbrace{\int_{f(\xi_t)\geq K_2} f(\xi_t) dt}_{\geq K_2\mu_T} + \underbrace{\int_{f(\xi_t)\geq K_2} (X_t^\xi)^\top \Sigma X_t^\xi dt}_{\leq K_0\mu_T} \\
&\qquad + \alpha \frac{\mu_T}{T-\mu_T} \underbrace{\int_{f(\xi_t)< K_2} f(\xi_t) dt}_{\leq K_1 - K_2\mu_T} - \frac{\mu_T}{T-\mu_T} \underbrace{\int_{f(\xi_t)< K_2} (X_t^\xi)^\top \Sigma X_t^\xi dt}_{\geq 0}.
\end{aligned}
$$

If K_2 is large enough, the right hand side of the above equation is smaller than zero for all possible values $\mu_T \in]0, \frac{K_1}{K_2}]$, which contradicts the optimality of $\xi = \xi^{(1)}$. This completes the proof that ξ is essentially bounded. Note that a suitably large bound K_2 holds for all time horizons longer than T and for all initial portfolios that are close to X_0, establishing the uniform boundedness of ξ_0.

Since ξ is essentially bounded, we can apply the Theorems of Tonelli and Weierstrass (see Cesari (1983), Theorem 2.6) and find that X_t^ξ is C^1 everywhere and C^2 until it attains zero. Furthermore, it fulfils the Euler-Lagrange equation. Bellman's principle of optimality for the optimal trading vector ξ follows by the additivity of mean costs and variance of proceeds, as already noted by Almgren and Chriss (2001). □

Proposition A.3. *For an* infinite *liquidation time horizon, a mean-variance optimal liquidation strategy* $\bar\xi^{(X_0)}$ *exists for all initial portfolios* $X_0 \in \mathbb{R}^n$. *The portfolio evolution* $X_t^{\bar\xi^{(X_0)}}$ *is* C^1 *in* t *(i.e., the optimal trading vector* $\bar\xi_t^{(X_0)}$ *is continuous). We denote the time at which the portfolio* X_t *attains zero by*

$$T_0 := \inf\{t > 0 : X_t^{\bar\xi} = 0\} \in]0, \infty].$$

For $t \in [0, T_0[$, *the portfolio evolution* X_t^ξ *is* C^2 *and fulfils the Euler-Lagrange equation*

$$\Sigma X_t = D^2 f(-\dot X_t) \ddot X_t. \tag{29}$$

The optimal trading vector $\bar\xi^{(X_0)}$ *satisfies Bellman's principle of optimality, i.e.,*

$$\bar\xi_t^{(X_0)} = \bar\xi_0^{(X_t)} =: \bar a(X_t).$$

with a continuous vector field $\bar a : X \in \mathbb{R}^n \to \bar a(X) \in \mathbb{R}^n$.

Proof. First, we introduce some shorthand notation. For a sequence $(\mathrm{X}_0^{(i)}, T^{(i)}) \in \mathbb{R}^n \times \mathbb{R}$, we define

$$\begin{aligned}
\xi^{(i)} &:= \xi^{(\mathrm{X}_0^{(i)}, T^{(i)})} \\
\mathrm{X}_t^{(i)} &:= \mathrm{X}_t^{\xi^{(i)}} \\
T_0^{(i)} &:= \inf\{t > 0 : \mathrm{X}_t^{(i)} = 0\} \in]0, T^{(i)}].
\end{aligned}$$

In Proposition A.2, we established the uniform boundedness of ξ_0. For each $\mathrm{X}_0 \in \mathbb{R}^n$, we can therefore select a sequence $(\mathrm{X}_0^{(i)}, T^{(i)})$ with $\lim_{i\to\infty} \mathrm{X}_0^{(i)} = \mathrm{X}_0$ and $\lim_{i\to\infty} T^{(i)} = \infty$ such that $\xi_0^{(i)}$ converges to $\lim_{i\to\infty} \xi_0^{(i)} =: \xi_0^{(\infty)}$. Then we define $\mathrm{X}_t^{(\infty)}$ as the solution to the Euler-Lagrange equation with initial values $\mathrm{X}_0^{(\infty)} = \mathrm{X}_0$ and $\dot{\mathrm{X}}_0^{(\infty)} = -\xi_0^{(\infty)}$ until $\mathrm{X}_t^{(\infty)}$ attains zero at time $T_0^{(\infty)} \in]0, \infty]$. On $[T_0^{(\infty)}, \infty[$, we define $\mathrm{X}_t^{(\infty)} \equiv 0$.

Let $\phi(t, \mathrm{X}_0, \xi_0)$ be the position and trading speed $(\mathrm{X}_t, \dot{\mathrm{X}}_t)$ at time t of the solution to the Euler-Lagrange equation with initial values X_0 and ξ_0. Since the Euler-Lagrange equation is Lipschitz continuous on $\mathrm{X} \neq 0$, for any compact subset $[0, T] \subset [0, T_0^{(\infty)}[$ there exists an open set O with $(\mathrm{X}_0, \xi_0) \in O \subset \mathbb{R}^{2n}$ such that ϕ is continuous on $[0, T] \times O$. On any compact subset of $[0, T] \times O$, this ensures uniform continuity. Since $(\mathrm{X}^{(i)})$ is a family of solutions to the Euler-Lagrange equation with converging initial values $(\mathrm{X}_0^{(i)}, \xi_0^{(i)})$, this implies uniform convergence of $(\mathrm{X}^{(i)}, \dot{\mathrm{X}}^{(i)})$ to $(\mathrm{X}^{(\infty)}, \dot{\mathrm{X}}^{(\infty)})$ on any compact subset $[0, T] \subset [0, T_0^{(\infty)}[$. Therefore

$$\begin{aligned}
\int_0^\infty (f(\xi_s^{(\infty)}) + (\mathrm{X}_s^{(\infty)})^\top \Sigma \mathrm{X}_s^{(\infty)}) ds &= \lim_{T \to T_0^{(\infty)}} \int_0^T (f(\xi_s^{(\infty)}) + (\mathrm{X}_s^{(\infty)})^\top \Sigma \mathrm{X}_s^{(\infty)}) ds \\
&= \lim_{T \to T_0^{(\infty)}} \lim_{i\to\infty} \int_0^T (f(\xi_s^{(i)}) + (\mathrm{X}_s^{(i)})^\top \Sigma \mathrm{X}_s^{(i)}) ds \\
&\le \lim_{i\to\infty} \int_0^\infty (f(\xi_s^{(i)}) + (\mathrm{X}_s^{(i)})^\top \Sigma \mathrm{X}_s^{(i)}) ds.
\end{aligned}$$

This establishes that $\mathrm{X}^{(\infty)}$ is "at least as good" as the limit of the finite time strategies $\mathrm{X}^{(i)}$. In Proposition A.6, we show that $\mathrm{X}^{(\infty)} \in \bar{\mathcal{X}}$. We now show that no strategy can be any better than this limit. Let $\mathrm{X}^{[\infty]} \in \bar{\mathcal{X}}$ be a deterministic admissible strategy with $\mathrm{X}_0^{[\infty]} = \mathrm{X}_0$ and finite mean-variance cost

$$\int_0^\infty (f(\xi_s^{[\infty]}) + (\mathrm{X}_s^{[\infty]})^\top \Sigma \mathrm{X}_s^{[\infty]}) ds < \infty.$$

Then $\mathrm{X}_t^{[\infty]}$ converges to zero as t tends to infinity. We define a sequence of trading strategies $\mathrm{X}^{[i]}$ that liquidate the portfolio $\mathrm{X}_0^{(i)}$ by time $T^{(i)} > 2$ in the following way:

$$\mathrm{X}_t^{[i]} := \begin{cases} \mathrm{X}_t^{[\infty]} + (1-t)(\mathrm{X}_0^{(i)} - \mathrm{X}_0) & \text{for } 0 \le t \le 1 \\ \mathrm{X}_t^{[\infty]} & \text{for } 1 < t < T^{(i)} - 1 \\ (T^{(i)} - t)\mathrm{X}_{T^{(i)}-1}^{[\infty]} & \text{for } T^{(i)} - 1 \le t \le T^{(i)} \\ 0 & \text{for } t > T^{(i)}. \end{cases}$$

We then have

$$\begin{aligned}
\lim_{i\to\infty} \int_0^\infty (f(\xi_s^{(i)}) + (\mathrm{X}_s^{(i)})^\top \Sigma \mathrm{X}_s^{(i)}) ds &\le \lim_{i\to\infty} \int_0^\infty (f(\xi_s^{[i]}) + (\mathrm{X}_s^{[i]})^\top \Sigma \mathrm{X}_s^{[i]}) ds \\
&= \lim_{T\to\infty} \lim_{i\to\infty} \int_0^T (f(\xi_s^{[i]}) + (\mathrm{X}_s^{[i]})^\top \Sigma \mathrm{X}_s^{[i]}) ds \\
&= \int_0^\infty (f(\xi_s^{[\infty]}) + (\mathrm{X}_s^{[\infty]})^\top \Sigma \mathrm{X}_s^{[\infty]}) ds.
\end{aligned}$$

Hence $\mathrm{X}^{(\infty)}$ is mean-variance optimal. Because it is unique by Lemma A.1, we see that $\xi_0^{(i)}$ converges to the same vector ξ_0^∞ for any sequence $(\mathrm{X}_0^{(i)}, T^{(i)})$. Therefore $\xi_0^{(\infty)}$ depends continuously on X_0. The validity of the Euler-Lagrange equation carries over by construction; Bellman's principle of optimality follows again by the additivity of mean costs and variance of proceeds. ☐

The next proposition establishes a special form of the identity established by Beltrami (1868) and rediscovered by Hilbert in 1900; see also Bolza (1909)[pp. 107].

Proposition A.4. *The vector field* $\bar{\mathrm{a}}$ *fulfils*

$$\frac{f(\bar{\mathrm{a}}(\mathrm{X}))}{\mathrm{X}^\top \Sigma \mathrm{X}} = \frac{1}{2\alpha} \text{ for all } \mathrm{X} \in \mathbb{R}^n \backslash \{0\}.$$

Proof. Let X_t be a mean-variance optimal strategy. Then

$$\begin{aligned} \frac{d}{dt}\left(f(-\dot{\mathrm{X}}_t) + \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t\right) &= -\nabla f(-\dot{\mathrm{X}}_t)\ddot{\mathrm{X}}_t + \mathrm{X}_t^\top \Sigma \dot{\mathrm{X}}_t \\ &= -\nabla f(-\dot{\mathrm{X}}_t)\ddot{\mathrm{X}}_t + (\ddot{\mathrm{X}}_t)^\top D^2 f(-\dot{\mathrm{X}}_t)\dot{\mathrm{X}}_t \qquad (30) \\ &= \frac{d}{dt}(-\nabla f(-\dot{\mathrm{X}}_t)\dot{\mathrm{X}}_t) \\ &= \frac{d}{dt}((\alpha+1) f(-\dot{\mathrm{X}}_t)) \qquad (31) \end{aligned}$$

where Equation (30) follows by the Euler-Lagrange equation (29) and Equation (31) by the scaling property (1) which implies

$$\nabla f(\mathrm{a})\mathrm{a} = \lim_{s\to 0} \frac{f((1+s)\mathrm{a}) - f(\mathrm{a})}{s} = (\alpha+1) f(\mathrm{a}). \qquad (32)$$

Hence

$$-\alpha f(\bar{\mathrm{a}}(X_0)) + \frac{1}{2}\mathrm{X}_0^\top \Sigma \mathrm{X}_0 = \lim_{t\to 0}\left(-\alpha f(\bar{\mathrm{a}}(X_t)) + \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t\right) = 0.$$

The desired equality follows immediately. □

Finally, we show that the mean-variance value function fulfils the dynamic programming PDE.

Proposition A.5. *The mean-variance value function*

$$\bar{v}(\mathrm{X}_0) := \inf_{\bar{\xi} \in \bar{\mathcal{X}}} \left[\int_0^\infty \left(f(\bar{\xi}_s) + \frac{1}{2}(\mathrm{X}_s^{\bar{\xi}})^\top \Sigma \mathrm{X}_s^{\bar{\xi}}\right) ds\right]$$

is C^1 *and fulfils*

$$\nabla f(\bar{\mathrm{a}}(\mathrm{X})) = \bar{v}_{\mathrm{X}}. \qquad (33)$$

Proof. The mean-variance value function is convex because of the convexity of $f(\xi) + \frac{1}{2}\mathrm{X}^\top \Sigma \mathrm{X}$. The function $\bar{v}$ is therefore necessarily differentiable at $\mathrm{X}_0 \in \mathbb{R}^n$, if it is bounded from above by a smooth function $\tilde{v}$ that touches $\bar{v}$ at X_0, i.e., $\tilde{v}(\mathrm{X}_0) = \bar{v}(\mathrm{X}_0)$. Such a function $\tilde{v}$ however can be constructed as

$$\tilde{v}(\mathrm{X}) = \int_0^\infty \left(f(\xi_t^{\mathrm{X}}) + (\mathrm{X}_t^{\xi^{\mathrm{X}}})^\top \Sigma \mathrm{X}_t^{\xi^{\mathrm{X}}}\right) dt$$

with

$$\xi_t^{\mathrm{X}} := \bar{\xi}_t^{(\mathrm{X}_0)} + M_t(\mathrm{X} - \mathrm{X}_0)$$

where

$$M_t := (\bar{\xi}_t^{(\mathrm{X}_0+\mathrm{e}_1)} - \bar{\xi}_t^{(\mathrm{X}_0)}, \bar{\xi}_t^{(\mathrm{X}_0+\mathrm{e}_2)} - \bar{\xi}_t^{(\mathrm{X}_0)}, \ldots, \bar{\xi}_t^{(\mathrm{X}_0+\mathrm{e}_n)} - \bar{\xi}_t^{(\mathrm{X}_0)}) \in \mathbb{R}^{n\times n}$$

and e_i is the ith unit vector. Therefore $\bar{v}$ is differentiable. By the dynamic programming principle, we have that for any absolutely continuous path $\mathrm{X} : \mathbb{R}_0^+ \to \mathbb{R}^n$:

$$\bar{v}(\mathrm{X}_0) \leq \bar{v}(\mathrm{X}_t) + \int_0^t \left(f(-\dot{\mathrm{X}}_s) + \frac{1}{2}\mathrm{X}_s^\top \Sigma \mathrm{X}_s\right) ds$$

with equality for the optimal strategy $\mathrm{X}^{\bar{\xi}}$. Since $\bar{v}$ is differentiable, this implies

$$0 \leq \bar{v}_{\mathrm{X}}(\mathrm{X}_0)\dot{\mathrm{X}}_0 + f(-\dot{\mathrm{X}}_0) + \frac{1}{2}\mathrm{X}_0^\top \Sigma \mathrm{X}_0.$$

The right hand side therefore attains its minimum at the optimal $\dot{\mathrm{X}}_0 = -\bar{\mathrm{a}}(\mathrm{X}_0)$ and therefore

$$\nabla f(\bar{\mathrm{a}}(\mathrm{X}_0)) = \bar{v}_{\mathrm{X}}(\mathrm{X}_0).$$

This establishes Equation (33) and that the mean-variance cost $\bar{v}$ is C^1. □

Proposition A.6. *For any* $\mathrm{X}_0 \in \mathbb{R}^n$*, the deterministic mean-variance optimal trading strategy* $\bar{\xi} = \bar{\xi}^{(\mathrm{X}_0)}$ *satisfies*

$$\lim_{t\to\infty} (\mathrm{X}_t^{\bar{\xi}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}} t \ln\ln t = 0. \tag{34}$$

It is hence an admissible trading strategy, i.e., $\bar{\xi} \in \bar{\mathcal{X}} \subset \mathcal{X}$*.*

For the proof, we need the following lemma.

Lemma A.7. *Let* $\mathrm{Y}_0 = r\mathrm{X}_0$ *and let* X *and* Y *be the corresponding mean-variance optimal strategies. Then we have that*

$$\mathrm{Y}_t = r\mathrm{X}_{bt} \text{ with } b := r^{\frac{1-\alpha}{1+\alpha}}.$$

Proof of Lemma A.7. Let us define

$$\hat{\mathrm{X}}_t := \frac{1}{r}\mathrm{Y}_{\frac{t}{b}}$$
$$\hat{\mathrm{Y}}_t := r\mathrm{X}_{bt}.$$

Then $\hat{\mathrm{X}}$ and $\hat{\mathrm{Y}}$ are deterministic strategies with $\hat{\mathrm{X}}_0 = \mathrm{X}_0$ and $\hat{\mathrm{Y}}_0 = \mathrm{Y}_0$, and we obtain

$$\begin{aligned}
\bar{v}(\mathrm{X}_0) &\le \int_0^\infty \left(f(-\dot{\hat{\mathrm{X}}}_s) + \frac{1}{2}\hat{\mathrm{X}}_s^\top \Sigma \hat{\mathrm{X}}_s \right) ds \\
&= \left(\frac{1}{rb}\right)^{\alpha+1} b \int_0^\infty f(-\dot{\mathrm{Y}}_s) ds + \left(\frac{1}{r}\right)^2 b \int_0^\infty \frac{1}{2}\mathrm{Y}_s^\top \Sigma \mathrm{Y}_s ds \\
&= r^{-\frac{3\alpha+1}{\alpha+1}} \bar{v}(\mathrm{Y}_0) \\
&\le r^{-\frac{3\alpha+1}{\alpha+1}} \int_0^\infty \left(f(-\dot{\hat{\mathrm{Y}}}_s) + \frac{1}{2}\hat{\mathrm{Y}}_s^\top \Sigma \hat{\mathrm{Y}}_s \right) ds \\
&= r^{-\frac{3\alpha+1}{\alpha+1}} \left((rb)^{\alpha+1} \frac{1}{b} \int_0^\infty f(-\dot{\mathrm{X}}_s) ds + r^2 \frac{1}{b} \int_0^\infty \frac{1}{2}\mathrm{X}_s^\top \Sigma \mathrm{X}_s ds \right) \\
&= \bar{v}(\mathrm{X}_0).
\end{aligned}$$

All the inequalities above are thus equalities, and hence $\hat{\mathrm{X}}$ and $\hat{\mathrm{Y}}$ are optimal. The lemma follows since the optimal strategies are unique. □

Proof of Proposition A.6. It is clear that $\bar{\xi}^{(\mathrm{X}_0)}$ satisfies the conditions of Section 2. To see that it is admissible, i.e., that , i.e., that $\bar{\xi}^{(\mathrm{X}_0)} \in \mathcal{X}$, the only thing left to prove is Equation (34). First, we observe that by Lemma A.7, it is sufficient to prove this equation for X_0 with $\mathrm{X}_0^\top \Sigma \mathrm{X}_0 = 1$. Let us first define

$$\tau_0 := \sup_{\mathrm{X}_0 \text{ with } \mathrm{X}_0^\top \Sigma \mathrm{X}_0 = 1} \max \left\{ t > 0 : (\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}} \ge \frac{1}{2} \right\}.$$

This τ_0 is the time it takes at most until $\mathrm{X}_0^\top \Sigma \mathrm{X}_0$ is reduced from 1 to $\frac{1}{2}$. By Lemma A.7, we obtain that

$$\tau_1 := \sup_{\mathrm{X}_0 \text{ with } \mathrm{X}_0^\top \Sigma \mathrm{X}_0 = \frac{1}{2}} \max \left\{ t > 0 : (\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}} \ge \frac{1}{4} \right\} = 2^{\frac{1-\alpha}{1+\alpha}} \tau_0$$

or more generally

$$\tau_k := \sup_{\mathrm{X}_0 \text{ with } \mathrm{X}_0^\top \Sigma \mathrm{X}_0 = \left(\frac{1}{2}\right)^k} \max \left\{ t > 0 : (\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}} \ge \left(\frac{1}{2}\right)^{k+1} \right\} = 2^{k\frac{1-\alpha}{1+\alpha}} \tau_0.$$

Let $\mathrm{X}_0 \in \mathbb{R}^n$ with $\mathrm{X}_0^\top \Sigma \mathrm{X}_0 = 1$. Then for all $t \ge \sum_0^k \tau_k$, we have that

$$(\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}} \le \left(\frac{1}{2}\right)^{k+1}.$$

For $\alpha \ge 1$, we have that $\tau_k \le \tau_0$; $(\mathrm{X}_t^{\bar{\xi}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}}$ is therefore bounded from above by an exponential function. For $0 < \alpha < 1$, we see that $(\mathrm{X}_t^{\bar{\xi}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}}$ is bounded from above by $K(t+1)^{\frac{\alpha+1}{\alpha-1}}$ for a $K > 0$. In both cases we see that Equation (34) holds. □

A.2 Existence and characterization of a smooth solution of the HJB equation

As a first step, we observe that $\lim_{M\to\infty} u(M) < \infty$ due to the boundedness of the risk aversion, and we can thus assume without loss of generality that

$$\lim_{M\to\infty} u(M) = 0.$$

Proposition A.8. *There exists a smooth ($C^{2,4}$) solution of*

$$\tilde{a}_Y = -\frac{2\alpha+1}{\alpha+1}\tilde{a}^\alpha \tilde{a}_M + \frac{\alpha(\alpha-1)}{\alpha+1}\left(\frac{\tilde{a}_M}{\tilde{a}}\right)^2 + \frac{\alpha}{\alpha+1}\frac{\tilde{a}_{MM}}{\tilde{a}} \tag{35}$$

with initial value

$$\tilde{a}(0, M) = A(M)^{\frac{1}{\alpha+1}}. \tag{36}$$

The solution satisfies

$$\tilde{a}_{min} := \inf_{M\in\mathbb{R}} A(M)^{\frac{1}{\alpha+1}} \le \tilde{a}(Y, M) \le \sup_{M\in\mathbb{R}} A(M)^{\frac{1}{\alpha+1}} =: \tilde{a}_{max}. \tag{37}$$

The function $\tilde{a}$ is $C^{2,4}$ in the sense that it has a continuous derivative $\frac{\partial^{i+j}}{\partial Y^i \partial M^j}\tilde{a}(Y, M)$ if $2i + j \le 4$. In particular, $\tilde{a}_{YMM}$ and $\tilde{a}_{MMM}$ exist and are continuous. We do not establish the uniqueness of $\tilde{a}$ directly in the preceding proposition. However, it follows from Proposition A.16.

The statement follows from the following auxiliary theorem from the theory of parabolic partial differential equations. We do not establish the uniqueness of $\tilde{a}$ directly in the preceding proposition. However, it follows from Proposition A.16.

Theorem A.9 (Auxiliary theorem: Solution of Cauchy problem)**.** *There is a smooth solution ($C^{2,4}$)*

$$g : (t, x) \in \mathbb{R}_0^+ \times \mathbb{R} \to g(t, x) \in \mathbb{R}$$

for the parabolic partial differential equation[13]

$$g_t - \frac{d}{dx}\kappa(x, t, g, g_x) + \theta(x, t, g, g_x) = 0 \tag{38}$$

with initial value condition

$$g(0, x) = \psi_0(x)$$

if all of the following conditions are satisfied:

- *$\psi_0(x)$ is smooth (C^4) and bounded*
- *κ and θ are smooth (C^3 respectively C^2)*
- *There are constants b_1 and $b_2 \ge 0$ such that for all x and u:*

$$\left(\theta(x, t, u, 0) - \frac{\partial\kappa}{\partial x}(x, t, u, 0)\right) u \ge -b_1 u^2 - b_2.$$

- *For all $M > 0$, there are constants $\mu_M \ge \nu_M > 0$ such that for all x, t, u and p that are bounded in modulus by M:*

$$\nu_M \le \frac{\partial\kappa}{\partial p}(x, t, u, p) \le \mu_M$$

  and

$$\left(|\kappa| + \left|\frac{\partial\kappa}{\partial u}\right|\right)(1 + |p|) + \left|\frac{\partial\kappa}{\partial x}\right| + |\theta| \le \mu_M(1 + |p|)^2.$$

[13]Here, g_t refers to $\frac{d}{dt}g$ and not $g(t)$.

Proof. The theorem is a direct consequence of Theorem 8.1 in Chapter V of Ladyzhenskaya, Solonnikov, and Ural'ceva (1968). In the following, we outline the last step of its proof because we will use it for the proof of subsequent propositions.

The conditions of the theorem guarantee the existence of solutions g_N of Equation (38) on the strip $\mathbb{R}_0^+ \times [-N, N]$ with boundary conditions

$$g_N(0, x) = \psi_0(x) \text{ for all } x \in [-N, N]$$

and

$$g_N(t, \pm N) = \psi_0(\pm N) \text{ for all } t \in \mathbb{R}_0^+.$$

These solutions converge smoothly as N tends to infinity: $\lim_{N\to\infty} g_N = g$. □

Proof of Proposition A.8. We want to apply Theorem A.9 and set

$$\begin{aligned}
\kappa(x, t, u, p) &:= h_1(u)p \\
\theta(x, t, u, p) &:= h_2(u)p - h_3(u)p^2 + h_1'(u)p^2 \\
\psi_0(x) &:= A(M)^{\frac{1}{\alpha+1}}
\end{aligned}$$

with smooth functions $h_1, h_2, h_3 : \mathbb{R} \to \mathbb{R}$. With

$$h_1(u) = \frac{\alpha}{(\alpha+1)u} \qquad h_2(u) = \frac{2\alpha+1}{(\alpha+1)}u^\alpha \qquad h_3(u) = \frac{\alpha(\alpha-1)}{(\alpha+1)u^2}, \tag{39}$$

Equation (38) becomes Equation (35) by relabeling the coordinates from t to Y and from x to M. All conditions of Theorem A.9 are fulfilled, if we take h_1, h_2 and h_3 to be smooth non-negative functions bounded away from zero and infinity and fulfilling Equation (39) for $\tilde{a}_{min} \le u \le \tilde{a}_{max}$. With these functions, there exists a smooth solution to

$$g_t = -h_2(g)g_x + h_3(g)g_x^2 + h_1(g)g_{xx}.$$

We now show that this solution g also fulfils

$$g_t = -\frac{2\alpha+1}{\alpha+1}g^\alpha g_x + \frac{\alpha(\alpha-1)}{\alpha+1}\left(\frac{g_x}{g}\right)^2 + \frac{\alpha}{\alpha+1}\frac{g_{xx}}{g}$$

by using the maximum principle to show that $\tilde{a}_{min} \le g \le \tilde{a}_{max}$. First assume that there is a (t_0, x_0) such that $g(t_0, x_0) > \tilde{a}_{max}$. Then there is an $N > 0$ and $\gamma > 0$ such that also $\tilde{g}_N(t_0, x_0) := g_N(t_0, x_0)e^{-\gamma t_0} > \tilde{a}_{max}$ with g_N as constructed in the proof of Theorem A.9. Then $\max_{t\in[0,t_0],x\in[-N,N]} \tilde{g}_N(t, x)$ is attained at an interior point (t_1, x_1), i.e., $0 < t_1 \le t_0$ and $-N < x_1 < N$. We thus have

$$\begin{aligned}
\tilde{g}_{N,t}(t_1, x_1) &\ge 0 \\
\tilde{g}_{N,x}(t_1, x_1) &= 0 \\
\tilde{g}_{N,xx}(t_1, x_1) &\le 0.
\end{aligned}$$

We furthermore have that

$$\begin{aligned}
\tilde{g}_{N,t} &= e^{-\gamma t}g_{N,t} - \gamma e^{-\gamma t}g_N \\
&= -e^{-\gamma t}h_2(g_N)g_{N,x} + e^{-\gamma t}h_3(g_N)g_{N,x}^2 + e^{-\gamma t}h_1(g_N)g_{N,xx} - \gamma e^{-\gamma t}g_N \\
&= -h_2(g_N)\tilde{g}_{N,x} + h_3(g_N)\tilde{g}_{N,x}g_{N,x} + h_1(g_N)\tilde{g}_{N,xx} - \gamma\tilde{g}_N
\end{aligned}$$

and therefore that

$$\tilde{g}_N(t_1, x_1) \le 0.$$

This however contradicts $\tilde{g}_N(t_1, x_1) \ge \tilde{g}_N(t_0, x_0) \ge \tilde{a}_{max} > 0$.

By a similar argument, we can show that if there is a point (t_0, x_0) with $g(t_0, x_0) < \tilde{a}_{min}$, then the interior minimum (t_1, x_1) of a suitably chosen $\tilde{g}_N := e^{-\gamma t}(g_N - \tilde{a}_{max}) < 0$ satisfies $\tilde{g}_N(t_1, x_1) \ge 0$ and thus causes a contradiction. □

Proposition A.10. *There exists a $C^{2,4}$-solution $\tilde{w} : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ of the transport equation*

$$\tilde{w}_Y = -\tilde{a}^\alpha \tilde{w}_M \tag{40}$$

with initial value

$$\tilde{w}(0, M) = u(M).$$

The solution satisfies

$$0 \geq \tilde{w}(Y, M) \geq u(M - \tilde{a}^\alpha_{max} Y)$$

and is increasing in M and decreasing in Y.

Proof. The proof uses the method of characteristics. Consider the function

$$P : (Y, S) \in \mathbb{R}_0^+ \times \mathbb{R} \to P(Y, S) \in \mathbb{R}$$

satisfying the ODE

$$P_Y(Y, S) = \tilde{a}(Y, P(Y, S))^\alpha \tag{41}$$

with initial value condition $P(0, S) = S$. Since $\tilde{a}^\alpha$ is smooth and bounded, a solution of the above ODE exists for each fixed S. For every Y, $P(Y, \cdot)$ is a diffeomorphism mapping $\mathbb{R}$ onto $\mathbb{R}$ that has the same regularity as $\tilde{a}$, i.e., belongs to $C^{2,4}$. We define

$$\tilde{w}(Y, M) = u(S) \quad \text{iff} \quad P(Y, S) = M.$$

Then $\tilde{w}$ is a $C^{2,4}$-function satisfying the initial value condition. By definition, we have

$$\begin{aligned} 0 &= \frac{d}{dY}\tilde{w}(Y, P(Y, S)) \\ &= \tilde{w}_M(Y, P(Y, S))P_Y(Y, S) + \tilde{w}_Y(Y, P(Y, S)) \\ &= \tilde{w}_M(Y, P(Y, S))\tilde{a}(Y, P(Y, S))^\alpha + \tilde{w}_Y(Y, P(Y, S)). \end{aligned}$$

Therefore $\tilde{w}$ fulfils the desired partial differential equation. Since $\tilde{a} \leq \tilde{a}_{max}$, we know that $P_Y \leq \tilde{a}^\alpha_{max}$ and hence $P(Y, S) \leq S + Y\tilde{a}^\alpha_{max}$ and thus $\tilde{w}(Y, M) \geq u(M - \tilde{a}^\alpha_{max} Y)$.

The monotonicity statements in the proposition follow because the family of solutions of the ODE (41) do not cross and since $\tilde{a}$ is positive. □

Proposition A.11. *The function $w(X, M) := \tilde{w}(\bar{v}(\mathrm{X}), M)$ has continuous derivatives up to w_{XMM} and w_{MMMM}, and it solves the HJB equation*

$$\min_{\mathrm{a}} \left[-\frac{1}{2} w_{MM} \mathrm{X}^\top \Sigma \mathrm{X} + w_M f(\mathrm{a}) + w_{\mathrm{X}} \mathrm{a} \right] = 0. \tag{42}$$

The unique minimum is attained at

$$\mathrm{a}(X, M) := \tilde{a}(\bar{v}(\mathrm{X}), M)\bar{\mathrm{a}}(\mathrm{X}). \tag{43}$$

Note that w is not necessarily everywhere twice differentiable in X; the single asset case with $\alpha < 1$ is a counterexample (see Equation (10)).

Proof. Assume for the moment that

$$\tilde{a}^{\alpha+1} = -\frac{\tilde{w}_{MM}}{\tilde{w}_M}. \tag{44}$$

Then with $Y = \bar{v}(\mathrm{X})$:

$$\begin{aligned} 0 &= -\frac{1}{2}\mathrm{X}^\top \Sigma \mathrm{X} \tilde{w}_M \left(\frac{\tilde{w}_{MM}}{\tilde{w}_M} + \tilde{a}^{\alpha+1} \right) \\ &= -\frac{1}{2}\mathrm{X}^\top \Sigma \mathrm{X} \tilde{w}_M \left(\frac{\tilde{w}_{MM}}{\tilde{w}_M} + \frac{2\alpha f(\bar{\mathrm{a}})}{\mathrm{X}^\top \Sigma \mathrm{X}} \tilde{a}^{\alpha+1} \right) && (45) \\ &= -\frac{1}{2}\tilde{w}_{MM}\mathrm{X}^\top \Sigma \mathrm{X} - \alpha \tilde{w}_M f(\mathrm{a}) && (46) \\ &= \inf_{\mathrm{a}} \left[-\frac{1}{2} w_{MM} \mathrm{X}^\top \Sigma \mathrm{X} + w_M f(\mathrm{a}) + w_{\mathrm{X}} \mathrm{a} \right]. && (47) \end{aligned}$$

Equation (45) holds because of Theorem 4.1, Equation (46) because of the scaling property of f (Equation (1)), and Equation (47) again because of the scaling property of f as in Equation (32). Note that the minimizer a as in Equation (43) is unique since ∇f is injective due to the convexity of f.

We now show that Equation (44) is fulfilled for all M and $Y = \bar{v}(\mathrm{X})$. First, observe that it holds for $Y = 0$. For general Y, consider the following two equations:

$$\frac{d}{dY}\tilde{a}^{\alpha+1} = -(2\alpha+1)\tilde{a}^{2\alpha}\tilde{a}_M + \alpha(\alpha-1)\tilde{a}^{\alpha-2}\tilde{a}_M^2 + \alpha\tilde{a}^{\alpha-1}\tilde{a}_{MM} \tag{48}$$

$$-\frac{d}{dY}\frac{\tilde{w}_{MM}}{\tilde{w}_M} = \tilde{a}^{\alpha}\frac{d}{dM}\frac{\tilde{w}_{MM}}{\tilde{w}_M} + \alpha\tilde{a}^{\alpha-1}\tilde{a}_M\frac{\tilde{w}_{MM}}{\tilde{w}_M} + \alpha(\alpha-1)\tilde{a}^{\alpha-2}\tilde{a}_M^2 + \alpha\tilde{a}^{\alpha-1}\tilde{a}_{MM}. \tag{49}$$

The first of these two equations holds because of Equation (35) and the second one because of Equation (40). Now we have

$$\frac{d}{dY}\left(\tilde{a}^{\alpha+1} + \frac{\tilde{w}_{MM}}{\tilde{w}_M}\right) = -\tilde{a}^{\alpha}\frac{d}{dM}\left(\tilde{a}^{\alpha+1} + \frac{\tilde{w}_{MM}}{\tilde{w}_M}\right) - \alpha\tilde{a}^{\alpha-1}\tilde{a}_M\left(\tilde{a}^{\alpha+1} + \frac{\tilde{w}_{MM}}{\tilde{w}_M}\right).$$

Hence, the function $g(Y, M) := \tilde{a}^{\alpha+1} + \frac{\tilde{w}_{MM}}{\tilde{w}_M}$ satisfies the linear PDE

$$g_Y = -\tilde{a}^{\alpha} g_M - \alpha\tilde{a}^{\alpha-1}\tilde{a}_M g$$

with initial value condition $g(0, M) = 0$. One obvious solution to this PDE is $g(Y, M) \equiv 0$. By the method of characteristics this is the unique solution to the PDE, since $\tilde{a}$ and $\tilde{a}_M$ are smooth and hence locally Lipschitz. □

The next auxiliary lemma will prove useful in the following.

Lemma A.12 (Auxiliary Lemma). *There are positive constants c_1, c_2, c_3, c_4 and b such that*

$$\begin{array}{ccccc} u(M) & \geq & w(\mathrm{X}, M) & \geq & u(M)\exp(b\bar{v}(\mathrm{X})) \\ 0 & \leq & w_M(\mathrm{X}, M) & \leq & c_1 + c_2\exp(-c_3 M + c_4\bar{v}(\mathrm{X})) \end{array} \tag{50}$$

for all $(\mathrm{X}, M) \in \mathbb{R}^n \times \mathbb{R}$.

Proof of Lemma A.12. The left hand side of the first inequality follows by the boundary condition for w and the monotonicity of w with respect to X as established in Proposition A.10. Since the risk aversion of u is bounded from above by $\tilde{a}_{max}^{\alpha+1}$, we have

$$u(M - \Delta) \geq u(M)e^{\tilde{a}_{max}^{\alpha+1}\Delta} \text{ for } \Delta \geq 0 \tag{51}$$

and thus by Proposition A.10

$$w(\mathrm{X}, M) \geq u(M - \tilde{a}_{max}^{\alpha}\bar{v}(\mathrm{X})) \geq u(M)e^{\tilde{a}_{max}^{2\alpha+1}\bar{v}(\mathrm{X})}$$

which establishes the right hand side of the first inequality with $b = \tilde{a}_{max}^{2\alpha+1}$.

For the second inequality, we will show the equivalent inequality

$$0 \leq \tilde{w}_M(Y, M) \leq c_1 + c_2\exp(-c_3 M + c_4 Y).$$

The left hand side follows since $\tilde{w}$ is increasing in M by Proposition A.10. For the right hand side, note that also the “risk aversion” of $\tilde{w}$ is bounded by $\tilde{a}_{max}^{\alpha+1}$ due to Equation (44). Hence

$$\tilde{w}(Y, M_0) \geq \tilde{w}(Y, M) + \frac{\tilde{w}_M(Y, M)}{\tilde{a}_{max}^{\alpha+1}}\left(1 - e^{-\tilde{a}_{max}^{\alpha+1}(M_0 - M)}\right).$$

Since

$$\lim_{M_0\to\infty}\tilde{w}(Y, M_0) \leq \lim_{M_0\to\infty}u(M_0) = 0$$

we have

$$0 \geq \tilde{w}(Y, M) + \frac{\tilde{w}_M(Y, M)}{\tilde{a}_{max}^{\alpha+1}}$$

and thus

$$\tilde{w}_M(Y, M) \leq -\tilde{w}(Y, M)\tilde{a}_{max}^{\alpha+1} \leq -u(M - \tilde{a}_{max}^{\alpha}Y)\tilde{a}_{max}^{\alpha+1}.$$

Since u is bounded by an exponential function, we obtain the desired bound on $\tilde{w}_M$. □

A.3 Verification argument

We now connect the PDE results from Subsection A.2 with the optimal stochastic control problem introduced in Section 2. For any admissible strategy $\xi \in \mathcal{X}$ and $k \in \mathbb{N}$ we define

$$\tau_k^{\xi} := \inf\Big\{t \geq 0 \,\Big|\, \int_0^t f(\xi_s)\,ds \geq k\Big\}.$$

We proceed by first showing that $u(M_t^{\xi})$ and $w(\mathrm{X}_t^{\xi}, M_t^{\xi})$ fulfil local supermartingale inequalities. Thereafter we show that $w(\mathrm{X}_0, M_0) \geq \lim_{t\to\infty} \mathbb{E}[u(M_t^{\xi})]$ with equality for $\xi = \hat{\xi}$.

Lemma A.13. *For any admissible strategy ξ the expected utility $\mathbb{E}[\,u(M_t^{\xi})\,]$ is decreasing in t. Moreover, we have $\mathbb{E}[\,u(M_{t\wedge\tau_k^{\xi}}^{\xi})\,] \geq \mathbb{E}[\,u(M_t^{\xi})\,]$.*

Proof. Since $M_t^{\xi} - M_0$ is the difference of the true martingale $\int_0^t (\mathrm{X}_s^{\xi})^{\top}\sigma\,d\mathrm{B}_s$ and the increasing process $\int_0^t f(\xi_s)\,ds$, it satisfies the supermartingale inequality $\mathbb{E}[\,M_t^{\xi}\,|\,\mathcal{F}_s\,] \leq M_s^{\xi}$ for $s \leq t$ (even though it may fail to be a supermartingale due to the possible lack of integrability). Hence $\mathbb{E}[\,u(M_t^{\xi})\,]$ is decreasing according to Jensen's inequality.

For the second assertion, we first take $n = k$ and write for $\tau_m := \tau_m^{\xi}$

$$\mathbb{E}[\,u(M_{t\wedge\tau_k}^{\xi})\,] = \mathbb{E}\Big[\,u\Big(M_0 + \int_0^{t\wedge\tau_n} (\mathrm{X}_s^{\xi})^{\top}\sigma\,d\mathrm{B}_s - \int_0^{t\wedge\tau_k} f(\xi_s)\,ds\Big)\,\Big].$$

When sending n to infinity, the right-hand side decreases to

$$\mathbb{E}\Big[\,u\Big(M_0 + \int_0^{t} (\mathrm{X}_s^{\xi})^{\top}\sigma\,d\mathrm{B}_s - \int_0^{t\wedge\tau_k} f(\xi_s)\,ds\Big)\,\Big], \tag{52}$$

by dominated convergence because u is bounded from below by an exponential function, the integral of $f(\xi)$ is bounded by k, and the stochastic integrals are uniformly bounded from below by $\inf_{s\leq Kt} W_s$, where W is the DDS-Brownian motion of $\int (\mathrm{X}_s^{\xi})^{\top}\sigma\,d\mathrm{B}_s$ and K is an upper bound for $(\mathrm{X}^{\xi})^{\top}\Sigma\mathrm{X}^{\xi}$. Finally, the term in Equation (52) is clearly larger than or equal to $\mathbb{E}[\,u(M_t^{\xi})\,]$. ☐

Lemma A.14. *For any admissible strategy ξ, $w(\mathrm{X}_t^{\xi}, M_t^{\xi})$ is a local supermartingale with localizing sequence (τ_k^{ξ}).*

Proof. We use a verification argument similar to the ones in Schied, Schöneborn, and Tehranchi (2010) and Schied and Schöneborn (2009). For $T > t \geq 0$, Itô's formula yields that

$$\begin{aligned} w(\mathrm{X}_T^{\xi}, M_T^{\xi}) - w(\mathrm{X}_t^{\xi}, M_t^{\xi}) &= \int_t^T w_M(\mathrm{X}_s^{\xi}, M_s^{\xi})(\mathrm{X}_s^{\xi})^{\top}\sigma\,d\mathrm{B}_s \\ &\quad - \int_t^T \Big[w_M f(\xi_s) + w_{\mathrm{X}}\xi_s - \frac{1}{2}(\mathrm{X}_s^{\xi})^{\top}\Sigma\mathrm{X}_s^{\xi} w_{MM}\Big](\mathrm{X}_s^{\xi}, M_s^{\xi})\,ds. \end{aligned} \tag{53}$$

By Proposition A.11 the latter integral is non-negative and we obtain

$$w(\mathrm{X}_t^{\xi}, M_t^{\xi}) \geq w(\mathrm{X}_T^{\xi}, M_T^{\xi}) - \int_t^T w_M(\mathrm{X}_s^{\xi}, M_s^{\xi})(\mathrm{X}_s^{\xi})^{\top}\sigma\,d\mathrm{B}_s. \tag{54}$$

We will show next that the stochastic integral in Equation (54) is a local martingale with localizing sequence $(\tau_k) := (\tau_k^{\xi})$. For some constant C_1 depending on t, k, $|\sigma|$, M_0, and on the upper bound of $|\mathrm{X}^{\xi}|$ we have for $s \leq t \wedge \tau_k$

$$M_s^{\xi} = M_0 + (\mathrm{X}_s^{\xi})^{\top}\sigma\mathrm{B}_s + \int_0^s (\xi_q^{\top}\sigma\mathrm{B}_q - f(\xi_q))\,dq \geq -C_1\big(1 + \sup_{q\leq t}|\mathrm{B}_q|\big).$$

Using Lemma A.12, we see that for $s \leq t \wedge \tau_k$

$$0 \leq w_M(\mathrm{X}_s^{\xi}, M_s^{\xi}) \leq c_1 + c_2 \exp\Big(c_3 C_1\big(1 + \sup_{q\leq t}|\mathrm{B}_q|\big) + c_4 K^2\Big) \tag{55}$$

where K is the upper bound of $\bar{v}(\mathrm{X}^{\xi})$. Since $\sup_{q\leq t}|\mathrm{B}_q|$ has exponential moments of all orders, the martingale property of the stochastic integral in Equation (54) follows. Taking conditional expectations in Equation (54) thus yields the desired supermartingale property

$$w(\mathrm{X}^{\xi}_{t\wedge\tau_k}, M^{\xi}_{t\wedge\tau_k}) \geq \mathbb{E}[\, w(\mathrm{X}^{\xi}_{T\wedge\tau_k}, M^{\xi}_{T\wedge\tau_k})|\mathcal{F}_t\,]. \tag{56}$$

The integrability of $w(\mathrm{X}^{\xi}_{t\wedge\tau_k}, M^{\xi}_{t\wedge\tau_k})$ follows from Lemma A.12 and Equation (51) in a similar way as in Equation (55). □

Lemma A.15. *There is an adapted strategy* $\hat{\xi}$ *fulfilling*

$$\hat{\xi}_t = \mathrm{a}(\mathrm{X}^{\hat{\xi}}_t, M^{\hat{\xi}}_t) \tag{57}$$

with a *as defined in Equation 43. This* $\hat{\xi}$ *is admissible and satisfies* $\int_0^\infty f(\hat{\xi}_t)\,dt < K$ *for some constant* K*. Furthermore,* $w(\mathrm{X}^{\hat{\xi}}_t, M^{\hat{\xi}}_t)$ *is a martingale and*

$$w(\mathrm{X}_0, M_0) = \lim_{t\to\infty} \mathbb{E}[\, u(M^{\hat{\xi}}_t)\,] \leq v_2(\mathrm{X}_0, M_0) := \sup_{\xi\in\mathcal{X}} \lim_{t\to\infty} \mathbb{E}[u(M^{\xi}_t)].. \tag{58}$$

Proof. Consider the stochastic differential equation

$$d\begin{pmatrix} s_t \\ M_t \end{pmatrix} = \begin{pmatrix} \tilde{a}(\bar{v}(\mathrm{X}^{\bar{\xi}}_{s_t}), M_t)dt \\ -\tilde{a}(\bar{v}(\mathrm{X}^{\bar{\xi}}_{s_t}), M_t)^{\alpha+1} f(\bar{\mathrm{a}}(\mathrm{X}^{\bar{\xi}}_{s_t}))dt + (\mathrm{X}^{\bar{\xi}}_{s_t})^\top \sigma dB_t \end{pmatrix} \tag{59}$$

with initial condition $s_0 = 0$. The functions $\tilde{a}$ and $\bar{v}$ are differentiable, $\mathrm{X}^{\bar{\xi}}_s$ is differentiable in s, and by the Beltrami identity (9) we have

$$f(\bar{\mathrm{a}}(\mathrm{X}^{\bar{\xi}}_s)) = \frac{(\mathrm{X}^{\bar{\xi}}_s)^\top \Sigma \mathrm{X}^{\bar{\xi}}_s}{2\alpha}$$

which establishes that $f(\bar{\mathrm{a}}(\mathrm{X}^{\bar{\xi}}_s))$ is differentiable in s. Hence, Equation (59) satisfies local boundedness and Lipschitz conditions and hence has a solution; see for example Durrett (1996). We can now set $\hat{\mathrm{X}}_t := \mathrm{X}^{\bar{\xi}}_{s_t}$; the resulting stochastic process $\hat{\mathrm{X}}$ is absolutely continuous, and by setting $\hat{\xi}_t := -\dot{\hat{\mathrm{X}}}_t$ we obtain a solution of Equation (57) since $\hat{\xi}_t = -\dot{\mathrm{X}}^{\bar{\xi}}_{s_t}\dot{s}_t = \bar{\mathrm{a}}(\hat{\mathrm{X}}_t)\tilde{a}(\bar{v}(\hat{\mathrm{X}}_t), M_t) = \mathrm{a}(\hat{\mathrm{X}}_t, M_t)$. We observe that $\hat{\xi}$ is admissible if $\int_0^\infty f(\hat{\xi}_t)\,dt < K$ for some constant K; conditions (2) and (3) are clear by Proposition A.6 and the lower bound on $\tilde{a}$ (Proposition A.8). The upper bound for $\int_0^\infty f(\hat{\xi}_t)\,dt$ can be derived as follows:

$$\begin{aligned}\int_0^\infty f(\hat{\xi}_t)dt &= \int_0^\infty f(\tilde{a}(\bar{v}(\mathrm{X}^{\hat{\xi}}_t), M^{\hat{\xi}}_t)\bar{\mathrm{a}}(\mathrm{X}^{\hat{\xi}}_t))dt = \int_0^\infty \tilde{a}^{\alpha+1}(\bar{v}(\mathrm{X}^{\hat{\xi}}_t), M^{\hat{\xi}}_t) f(\bar{\mathrm{a}}(\mathrm{X}^{\hat{\xi}}_t))dt \\ &\leq \tilde{a}^{\alpha+1}_{max} \int_0^\infty f(\bar{\mathrm{a}}(\mathrm{X}^{\hat{\xi}}_t))dt \leq \frac{\tilde{a}^{\alpha+1}_{max}}{\tilde{a}_{min}} \bar{v}(\mathrm{X}_0).\end{aligned}$$

Next, with the choice $\xi = \hat{\xi}$ the rightmost integral in Equation (53) vanishes, and we get equality in Equation (56). Since $\tau^{\hat{\xi}}_K = \infty$, this proves the martingale property of $w(\mathrm{X}^{\hat{\xi}}_t, M^{\hat{\xi}}_t)$. Furthermore, we obtain from Equation (50) that

$$u(M^{\hat{\xi}}_t) \geq w(\mathrm{X}^{\hat{\xi}}_t, M^{\hat{\xi}}_t) \geq u(M^{\hat{\xi}}_t)\exp(b\bar{v}(\mathrm{X}^{\hat{\xi}}_t)).$$

Since $\bar{v}(\mathrm{X}^{\hat{\xi}}_t)$ uniformly converges to zero as t tends to infinity, we obtain Equation (58). □

Proposition A.16. *We have* $v_2 = w$*, and* $\hat{\xi}$ *of Lemma A.15 is the a.s. unique optimal strategy to achieve* v_2*.*

Proof. By Lemma A.15, we already have $w \leq v_2$. We now show that $v_2 \leq w$. Let ξ be any admissible strategy such that

$$\lim_{t\to\infty} \mathbb{E}[u(M^{\xi}_t)] > -\infty. \tag{60}$$

By Lemmas A.14 and A.12 we have for all k, t and $(\tau_k) := (\tau^{\xi}_k)$

$$w(\mathrm{X}_0, M_0) \geq \mathbb{E}[w(\mathrm{X}^{\xi}_{t\wedge\tau_k}, M^{\xi}_{t\wedge\tau_k})] \geq \mathbb{E}\left[u(M^{\xi}_{t\wedge\tau_k})\exp(b\bar{v}(\mathrm{X}^{\xi}_{t\wedge\tau_k}))\right].$$

As in the proof of Lemma A.13 one shows that

$$\begin{aligned}\liminf_{k\to\infty} \mathbb{E}\left[u(M^{\xi}_{t\wedge\tau_k})\exp(b\bar{v}(\mathrm{X}^{\xi}_{t\wedge\tau_k}))\right] &\geq \liminf_{k\to\infty} \mathbb{E}\left[u(M^{\xi}_{t})\exp(b\bar{v}(\mathrm{X}^{\xi}_{t\wedge\tau_k}))\right]\\ &= \mathbb{E}\left[u(M^{\xi}_{t})\exp(b\bar{v}(\mathrm{X}^{\xi}_{t}))\right].\end{aligned}$$

Hence,

$$w(\mathrm{X}_0, M_0) \geq \mathbb{E}[u(M^{\xi}_t)] + \mathbb{E}\left[u(M^{\xi}_t)(\exp(b\bar{v}(\mathrm{X}^{\xi}_t)) - 1)\right].$$

Let us assume for a moment that the second expectation on the right attains values arbitrarily close to zero. Then

$$w(\mathrm{X}_0, M_0) \geq \lim_{t\to\infty} \mathbb{E}[u(M^{\xi}_t)].$$

Taking the supremum over all admissible strategies ξ gives $w \geq v_2$. The optimality of $\hat{\xi}$ follows from Lemma A.15, its uniqueness from the fact that the functional $\mathbb{E}[u(M^{\xi}_t)]$ is strictly concave since u is concave and increasing and M^{ξ}_t is concave.

We now show that $\mathbb{E}\left[u(M^{\xi}_t)(\exp(b\bar{v}(\mathrm{X}^{\xi}_t)) - 1)\right]$ attains values arbitrarily close to zero. First we observe that

$$0 \geq u(M) \geq c_5 u_{MM}(M)$$

for a constant $c_5 > 0$, due to the boundedness of the risk aversion of u, and that

$$\exp(b\bar{v}(\mathrm{X}^{\xi}_t)) - 1 \leq c_6 b\bar{v}(\mathrm{X}^{\xi}_t),$$

due to the bound on X^{ξ}_t. Since X^{ξ}_t is uniformly bounded, we see that for every $\epsilon_1 > 0$ there is a $\epsilon_2 > 0$ such that the following bound holds uniformly:

$$\bar{v}(\mathrm{X}^{\xi}_t) < \epsilon_1 + \epsilon_2(\mathrm{X}^{\xi}_t)^{\top}\Sigma\mathrm{X}^{\xi}_t.$$

Combining the last three inequalities, we obtain

$$\begin{aligned} 0 &\geq \mathbb{E}\left[u(M^{\xi}_t)(\exp(b\bar{v}(\mathrm{X}^{\xi}_t)) - 1)\right]\\ &\geq bc_6\epsilon_1\mathbb{E}[u(M^{\xi}_t)] + bc_5c_6\epsilon_2\mathbb{E}[(\mathrm{X}^{\xi}_t)^{\top}\Sigma\mathrm{X}^{\xi}_t u_{MM}(M^{\xi}_t)]. \end{aligned} \tag{61}$$

Let us now assume that the second expectation of Equation (61) attains values arbitrarily close to zero. Then for each $\epsilon_1 > 0$ there is a $\tilde{t} \in \mathbb{R}^+$ such that

$$0 \geq \mathbb{E}\left[u(M^{\xi}_{\tilde{t}})(\exp(b\bar{v}(\mathrm{X}^{\xi}_{\tilde{t}})) - 1)\right] \geq bc_6\epsilon_1 \lim_{t\to\infty}\mathbb{E}[u(M^{\xi}_t)].$$

Sending ϵ_1 to zero yields that $\mathbb{E}\left[u(M^{\xi}_t)(\exp(b\bar{v}(\mathrm{X}^{\xi}_t)) - 1)\right]$ attains values arbitrarily close to zero, since $\lim_{t\to\infty}\mathbb{E}[u(M^{\xi}_t)]$ is bounded by assumption (see Equation (60)).

We finish the proof by showing that the second expectation of Equation (61) attains values arbitrarily close to zero. By Lemma A.13 and the same line of reasoning as in the proof of Lemma A.14, we have for all k, t and $(\tau_k) := (\tau^{\xi}_k)$

$$\begin{aligned} -\infty < \lim_{s\to\infty}\mathbb{E}[u(M^{\xi}_s)] &\leq \mathbb{E}[u(M^{\xi}_t)] \leq \mathbb{E}[u(M^{\xi}_{t\wedge\tau_k})]\\ &= u(M_0) + \mathbb{E}\left[\int_0^{t\wedge\tau_k} u_M(M^{\xi}_s)(\mathrm{X}^{\xi}_s)^{\top}\sigma\, d\mathrm{B}_s\right]\\ &\quad - \mathbb{E}\left[\int_0^{t\wedge\tau_k}\left[u_M f(\xi_s) - \frac{1}{2}(\mathrm{X}^{\xi}_s)^{\top}\Sigma\mathrm{X}^{\xi}_s u_{MM}\right](M^{\xi}_s)\, ds\right]\\ &= u(M_0) - \mathbb{E}\left[\int_0^{t\wedge\tau_k}\left[u_M f(\xi_s) - \frac{1}{2}(\mathrm{X}^{\xi}_s)^{\top}\Sigma\mathrm{X}^{\xi}_s u_{MM}\right](M^{\xi}_s)\, ds\right]. \end{aligned} \tag{62}$$

Sending k and t to infinity yields

$$\int_0^{\infty}\mathbb{E}\left[(\mathrm{X}^{\xi}_s)^{\top}\Sigma\mathrm{X}^{\xi}_s u_{MM}(M^{\xi}_s)\right] ds > -\infty \tag{63}$$

which concludes the proof. □

Proposition A.17. *We have $v = w$, and $\hat{\xi}$ of Lemma A.15 is the a.s. unique optimal strategy.*

Proof. For any admissible strategy ξ, the martingale

$$\int_0^t (\mathrm{X}_s)^\top \sigma d\mathrm{B}_s$$

is uniformly integrable due to the requirement in Equation (2). Therefore

$$\mathbb{E}[u(M_t^\xi)] \geq \mathbb{E}\left[u(M_\infty^\xi)\right]$$

follows as in the proof of Lemma A.13. Hence, Proposition A.16 yields

$$\mathbb{E}[u(M_\infty^\xi)] = \lim_{t\to\infty} \mathbb{E}[u(M_t^\xi)] \leq v_2(\mathrm{X}_0, M_0) \leq w(\mathrm{X}_0, M_0).$$

Taking the supremum over all admissible strategies ξ gives $v \leq w$. The converse inequality follows from Lemma A.14, since $\hat{\xi}$ is admissible. $\square$

Proof of Theorem 5.7. Fix $N > 0$ and let g_i denote the function $\tilde{g}_N$ constructed in the proof of Proposition A.8 when the parabolic boundary condition is given by $\tilde{g}_N(Y, M) = A_i(M)^{\frac{1}{\alpha+1}}$ for $Y = 0$ or $|M| = N$, where $i \in \{0; 1\}$. The result follows if we can show that $h := g_1 - g_0 \geq 0$. A straightforward computation shows that h solves the linear PDE

$$\begin{aligned}
h_Y =& \frac{2\alpha+1}{\alpha+1}\left(g_1^\alpha g_{1,M} - g_0^\alpha g_{0,M}\right) + \frac{\alpha(\alpha-1)}{\alpha+1}\left(\left(\frac{g_{1,M}}{g_1}\right)^2 - \left(\frac{g_{0,M}}{g_0}\right)\right) + \frac{\alpha}{\alpha+1}\left(\frac{g_{1,MM}}{g_1} - \frac{g_{0,MM}}{g_0}\right) \\
=& \frac{2\alpha+1}{\alpha+1}\left(g_1^\alpha h_M + \frac{g_1^\alpha - g_0^\alpha}{g_1 - g_0} g_{0,M} h\right) + \frac{\alpha(\alpha-1)}{\alpha+1}\left(\frac{g_0^2 g_{1,M} h_M + g_0^2 g_{0,M} h_M - g_0 g_{0,M}^2 h - g_1 g_{0,M}^2 h}{g_0^2 g_1^2}\right) \\
&+ \frac{\alpha}{\alpha+1}\left(\frac{g_0 h_{MM} - g_{0,MM} h}{g_0 g_1}\right) \\
=& \frac{1}{2} b_1 h_{MM} + b_2 h_M + V h,
\end{aligned}$$

where the coefficients b_1 and b_2 and the potential V are given by

$$\begin{aligned}
b_1 =& \frac{\alpha}{\alpha+1}\frac{1}{g_1}, \\
b_2 =& \frac{2\alpha+1}{\alpha+1} g_1^\alpha + \frac{\alpha(\alpha-1)}{\alpha+1}\frac{g_{1,M} + g_{0,M}}{g_1^2}, \\
V =& \frac{2\alpha+1}{\alpha+1}\frac{g_1^\alpha - g_0^\alpha}{g_1 - g_0} g_{0,M} - \frac{\alpha(\alpha-1)}{\alpha+1}\frac{(g_0 + g_1) g_{0,M}^2}{g_0^2 g_1^2} - \frac{\alpha}{\alpha+1}\frac{g_{0,MM}}{g_0 g_1}.
\end{aligned}$$

The parabolic boundary condition of h is

$$h(Y, M) = A_1^{\frac{1}{\alpha+1}} - A_0^{\frac{1}{\alpha+1}} =: \psi(M) \qquad \text{for } Y = 0 \text{ or } |M| = N. \tag{64}$$

The functions b_1, b_2, V, and ψ are smooth and (at least locally) bounded on $\mathbb{R}_+ \times [-N, N]$, and b_1 is bounded away from zero. Next, take $T > 0$, $M \in]-N, N[$, and let Z be the solution of the stochastic differential equation

$$dZ_t = \sqrt{b_1(T - t, Z_t)}\, dB_t + b_2(T - t, Z_t) dt, \qquad Z_0 = M, \tag{65}$$

which is defined up to time

$$\tau := \inf\left\{t \geq 0 \,\middle|\, |Z_t| = N \text{ or } t = T\right\}. \tag{66}$$

By a standard Feynman-Kac argument, h can then be represented as

$$h(T, M) = \mathbb{E}\Big[\psi(Z_\tau) \exp\Big(\int_0^\tau V(T - t, Z_t)\, dt\Big)\Big]. \tag{67}$$

Hence $h \geq 0$ as $\psi \geq 0$ by assumption. $\square$

Proof of Theorem 5.6. In Theorem 5.7 take $u^0(x) := u(x)$ and $u^1(x) := u(x + r)$. If u exhibits IARA, then $A^1 \geq A^0$ if $r > 0$ and hence $\tilde{a}^1 \geq \tilde{a}^0 = \tilde{a}$. But we clearly have $\tilde{a}^1(X, M) = \tilde{a}(X, M + r)$. The result for decreasing A follows by taking $r < 0$. $\square$

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